รายงานวิจัยฉบับสมบูรณ์

ชื่อโครงการ

Group Classification of one-dimensional nonisentropic equations of fluids with internal inertia

โดย

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EXECUTIVE SUMMARY

Many physical phenomena are described by Mathematical models, which are written through partial differential equations. Solutions of partial differential equations are used explain physical phenomena. Therefore, there is interest in finding exact solution of partial differential equations. Numerical methods are powerful, but they only give approximate solutions. Approximate solutions do not allow analyzing properties of equations. For studying properties of models one needs to know exact solutions of partial differential equations. The way to get exact solutions of partial differential equations is not easy.

One of methods for constructing exact solutions is group analysis. In the thesis we apply group analysis method to class of dispersive models. These models include a nonlinear one-velocity model of bubbly fluid at small volume (Iordanskii-Kogarko-Wingarden model) and the dispersive shallow water model (Green –Nagdi model)

Research objectives

The research is devoted to application of group analysis to one-dimensional nonisentropic equations of fluid with internal inertia. The objectives of research are follows.

1. To find equivalence Lie group
2. To find admitted Lie group

Scope and limitations of the study

The research will deal with equations of fluids where the function W depends on ρ, $\dot{\rho}$ and S

Research procedure

The research procedure to be used in this research consists of a number of steps which can briefly be described as follows.

1. Construct determining equations of the equivalence Lie group.
2. Construct determining equations of the admitted Lie group and solve them.

Expected results

The expect outcome of this research project is classification of one-dimensional nonisentropic equations of fluid with internal inertia, where the function W depends on ρ, $\dot{\rho}$ and S

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Abstract

In this research, a systematic application of the group analysis method for modeling fluids with internal inertia is presented. The equations studied include models such as the non-linear one-velocity model of a bubbly fluid (with incompressible liquid phase) at small volume concentration of gas bubbles (Iordanski (1960), Kogarko (1961), Wijngaarden (1968)), and the dispersive shallow water model (Green \& Naghdi (1976),Salmon (1988)). These models are obtained for special types of the potential function $W(\rho, \dot{\rho}, S)$ (Gavrilyuk \& Teshukov (2001)).The main feature of the present research is the study of the potential functions with $W_s \neq 0$. The group classification separates these models into 73 different classes.

บทคัดย่อ

งานวิจัยนี้ได้นำเสนอการประยุกต์วิธีการวิเคราะห์เชิงกลุ่มสำหรับการจำลองของไหลที่มีความเฉื่อยภายใน สมการที่ศึกษามีหลายรูปแบบเช่น แบบจำลองความเร็วหนึ่งมิติไม่เชิงเส้นของของไหลที่มีฟอง(กับของเหลวที่ไม่สามารถบีบอัดได้) ที่มีความเข้มข้นของฟองก๊าซที่มีปริมาตรน้อย (Iordanski (1960), Kogarko (1961), Wijngaarden (1968)), และแบบจำลองน้ำตื้นแบบกระจาย (Green &Naghdi (1976), Salmon (1988)) แบบจำลองเหล่านี้อยู่ในรูปฟังก์ชันศักย์ $W(\rho,\dot{\rho},S)$ งานวิจัยนี้ได้ศึกษาฟังก์ชันศักย์ที่ $W_S \neq 0$ การจำแนกประเภทเชิงกลุ่มแบ่งแบบจำลองเหล่านี้ออกเป็น 73 แบบที่แตกต่างกัน

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CHAPTER I

INTRODUCTION

Symmetry is a fundamental topic in many areas of physics and mathematics (Golubitsky and Stewart, 2002), (Marsden and Ratiu, 1994), (Olver, 1993).. Whereas group-theoretical methods play a prominent role in modern theoretical physics, a systematic use of them in constructing models of continuum mechanics has not been widely applied yet (Ovsiannikov, 1994). The present paper tries to help to fill this niche.

This manuscript is focused on group classification of a class of dispersive models (Gavrilyuk and Teshukov, 2001)*

$$\begin{gathered} \dot{\rho} + \rho\, div(u) = 0, \quad \rho\dot{u} + \nabla p = 0, \quad \dot{S} = 0, \\ p = \rho\frac{\delta W}{\delta \rho} - W = \rho(\frac{\partial W}{\partial \rho} - \frac{\partial}{\partial t}(\frac{\partial W}{\partial \dot{\rho}}) - div(\frac{\partial W}{\partial \dot{\rho}}u)) - W, \end{gathered} \tag{1.1}$$

where t is time, ∇ is the gradient operator with respect to space variables, ρ is the fluid density, u is the velocity field, $W(\rho, \dot{\rho}, S)$ is a given potential, "dot" denotes the material time derivative: $\dot{f} = \frac{df}{dt} = f_t + u\nabla f$ and $\frac{\delta W}{\delta \rho}$ denotes the variational derivative of W with respect to ρ at a fixed value of u. These models include the non-linear one-velocity model of a bubbly fluid (with incompressible liquid phase) at small volume concentration of gas bubbles (Ibragimov,1999), (Kogarko, 1961), (Wijngaarden,1968), and the dispersive shallow water model (Green & Naghdi (1975) , Salmon (1998)

Equations (4.1) were obtained in (Gavrilyuk and Teshukov, 2001), using

*See also references therein.

the Lagrangian of the form

$$L = \frac{1}{2}|u|^2 - W(\rho, \dot{\rho}, S).$$

This is an example of a medium behavior dependent not only on thermodynamical variables but also on their derivatives with respect to space and time. In this particular case the potential function depends on the total derivative of the density which reflects the dependence of the medium on its inertia. Another example of models where the medium behavior depends on the derivatives is constructed in (Gavrilyuk and Shugrin,1996) by assuming that the Lagrangian is of the form:

$$L = \frac{1}{2}|u|^2 - \varepsilon(\rho, |\nabla\rho|, S).$$

One of the methods for studying properties of differential equations is group analysis(Ovsiannikov, 1978), (Olver, 1986), (Ibragimov, 1999). This method is a basic method for constructing exact solutions of partial differential equations. A wide range of applications of group analysis to partial differential equations are collected in (Ibragimov, 1994), (Ibragimov, 1995), (Ibragimov, 1996). Group analysis, besides facilitating the construction of exact solutions, provides a regular procedure for mathematical modeling by classifying differential equations with respect to arbitrary elements. This feature of group analysis is the fundamental basis for mathematical modeling in the present paper.

An application of group analysis employs several steps. The first step is a group classification with respect to arbitrary elements. An algorithm of the group classification is applied in case where a system of differential equations has arbitrary elements in form of undefined parameters and functions. This algorithm is necessary since a specialization of the arbitrary elements can lead to an extension of admitted Lie groups. Group classification selects the functions $W(\rho, \dot{\rho}, S)$ such that the fluid dynamics equations (4.1) possess additional symmetry properties extending the kernel of admitted Lie groups. Algorithms of finding equivalence and admitted Lie groups are particular parts of the algorithm of the group classification.

A complete group classification of equations (4.1), where $W = W(\rho, \dot{\rho})$ is performed in (Hematulin and Meleshko and Gavrilyuk 2007), (one-dimensional case) and (Siriwat and Meleshko, 2008).(three-dimensional case). Invariant solutions of some particular cases which are separated out by the group classification are considered in (Hematulin and Meleshko and Gavrilyuk 2007),(Siriwat and Meleshko, 2008) . Group classification of the class of models describing the behavior of a dispersive continuum with $\varepsilon = \varepsilon(\rho, |\nabla\rho|)$ was studied in (Voraka and Meleshko, 2009). It is also worth to notice that the classical gas dynamics model corresponds to $W = W(\rho, S)$ (or $\varepsilon = \varepsilon(\rho, S)$). A complete group classification of the gas dynamics equations was presented in (Ovsiannikov, 1978). Later, an exhausted program of studying the models appeared in the group classification of the gas dynamics equations was announced in (Ovsiannikov, 1994). Some results of this program were summarized in (Ovsiannikov, 1999).

The present paper is focused on the group classification of the one-dimensional equations of fluids (4.1), where the function $W = W(\rho, \dot{\rho}, S)$ satisfies the conditions $W_{S\dot{\rho}\dot{\rho}} = 0$ and $W_S \neq 0$.

The paper is organized as follows. The next section studies the equivalence Lie group of transformations. The equivalence transformations are applied for simplifying the function $W(\rho, \dot{\rho}, S)$ in the process of the classification. In Section 3 the defining equations of the admitted Lie group are presented. Analysis of these equations separates equations (4.1) into equivalent classes. Notice that these classes are defined by the function $W(\rho, \dot{\rho}, S)$. For convenience of the reader, this analysis is split into two parts. A complete study of one particular case is given in

Section 4. Analysis of the other cases is similar but cumbersome. A complete study of the other cases is provided in Appendix. The result of the group classification of equations (4.1) where $W_{S\dot{\rho}\dot{\rho}} = 0$ and $W_S \neq 0$ is summarized in Table[1]. The admitted Lie algebras are also presented in this table.

CHAPTER II

FLUIDS WITH INTERNAL INERTIA

Equations of fluids with internal inertia are obtained on the base of the Euler-Lagrange principle with the Lagrangian

$$L = L(\rho, \rho_t, \nabla\rho, u),$$

where t is time, ∇ is the gradient operator with respect to the space variables x_1, x_2, x_3, ρ is the fluid density, $u = (u_1, u_2, u_3)$ is the velocity field. The density ρ and the velocity u satisfy the mass conservation equation and the equation of conservation of linear momentum

$$\dot{\rho} + \rho\, div(u) = 0, \quad \rho\dot{u} + \nabla p = 0, \tag{2.1}$$

where $\dot{()} = \partial/\partial t + u\nabla$ is the material derivative.

Among fluids with internal inertia two classes of models have been intensively studied. One class of models is constructed, assuming that the internal energy ε depends on the density ρ and the gradient of the density $|\nabla\rho|$. Review of these models can be found in (Gavrilyuk and Shugrin, 1996), (Anderson, McFadden and Wheeler, 1998) and references therein. The thesis is devoted to the study of another class of models. These models are obtained by assuming that the Lagrangian is of the form (Gavrilyuk and Teshukov, 2001):

$$L = \frac{1}{2}\rho|u|^2 - W(\rho, \dot{\rho}), \tag{2.2}$$

where $W(\rho, \dot{\rho})$ is a given potential. In this case the pressure p is given by the formula

$$p = \rho\left(\frac{\partial W}{\partial \rho} - \frac{\partial}{\partial t}(\frac{\partial W}{\partial \dot{\rho}}) - div(\frac{\partial W}{\partial \dot{\rho}}u)\right) - W. \tag{2.3}$$

Notice that if W is a linear function with respect to $\dot{\rho}$, then these equations are reduced to the classical Euler equations of a barotropic gas.

In the next sections we give examples of two the most well-known models.

2.1 Iordanski-Kogarko-Wijngaarden model

The Iordanski-Kogarko-Wijngaarden model describes a bubbly fluid with incompressible liquid phase and small volume concentration of gas bubbles. This type of model was proposed by Iordanski (1960), Kogarko (1961) and Wijngaarden (1968).

This mathematical model can be written in the form

$$\begin{aligned}
&\frac{\partial \rho_1}{\partial t} + div(\rho_1 u) = 0,\\
&\frac{\partial \rho_2}{\partial t} + div(\rho_2 u) = 0,\\
&\frac{\partial N}{\partial t} + div(N u) = 0,\\
&\dot{u} + \frac{1}{\rho}\nabla p = 0,\\
&R\ddot{R} + \frac{3}{2\dot{R}^2} = \frac{1}{\rho_{10}}(p_2 - p),
\end{aligned} \tag{2.4}$$

where

$$\rho_1 = \alpha_1 \rho_{10}, \quad \rho_2 = \alpha_2 \rho_{20}, \tag{2.5}$$

$\rho_{10} = const$ is the physical density of the liquid, ρ_{20} is the physical density of the gas, α_i, $(i = 1, 2)$ are the volume fractions: $\alpha_1 + \alpha_2 = 1$, N is the bubble number density, R is the bubble radius,

$$\rho = \rho_1 + \rho_2. \tag{2.6}$$

The volume fraction of the gas phase α_2 is defined by the formula

$$\alpha_2 = \frac{4}{3}\pi R^3 N. \tag{2.7}$$

The gas pressure p_2 is a given function of ρ_{20}:

$$p_2 = \rho_{20}^2 \varepsilon'_{20}(\rho_{20}),$$

where $\varepsilon_{20}(\rho_{20})$ is the internal energy of the gas phase. It is assumed that the mass concentrations $c_i = \rho_i/\rho$, $(i = 1, 2)$, and the number of bubbles per unit mass $n = N/\rho$ are constant. From (2.5)-(2.7) one obtains

$$R^3 = \frac{3}{4\pi n}(\frac{1}{\rho} - \frac{c_1}{\rho_{10}}), \ \ \rho_{20} = c_2(\frac{1}{\rho} - \frac{c_1}{\rho_{10}})^{-1}.$$

Bedford and Drumheller (1978) proved that equations (2.1), (2.3) can be obtained by using the potential function

$$W = \rho(C_2\varepsilon_{20}(\rho_{20}) - 2\pi n\rho_{10}R^3\dot{R}^2).$$

Replacing R and ρ_{20} in the potential function, one obtains that system of partial differential equations (2.4) is equivalent to (2.1) and (2.3) with the potential function

$$W(\rho, \dot{\rho}) = \psi(\rho) - k\dot{\rho}^2\rho^{\frac{8}{3}}(\frac{1}{\alpha - \rho})^{1/3},$$

where

$$\alpha = \frac{\rho_{10}}{c_1}, \qquad k = \frac{\rho_{10}}{8\pi n}(\frac{4\pi n\rho_{10}}{3c_1})^{1/3}.$$

2.2 Green-Naghdi model

Consider the dispersive shallow water equations of Green and Naghdi (1975)

$$\frac{\partial h}{\partial t} + div(hu) = 0 \tag{2.8}$$

$$\dot{u} + g\nabla h + \frac{\varepsilon^2}{(3h)}\nabla(h^2\ddot{h}) = 0, \tag{2.9}$$

where h is the water depth, u is the horizontal velocity, g is the gravity, ε is the ratio of the vertical length scale to the horizontal length scale. Replacing h by ρ, equations (2.8) take the form

$$\begin{aligned} &\frac{d\rho}{dt} + \rho div(u) = 0 \\ &\dot{u} + g\nabla\rho + \frac{\varepsilon^2}{3\rho}\nabla(\rho^2\ddot{\rho}) = 0. \end{aligned} \tag{2.10}$$

The last equation of (2.10) can be rewritten as

$$\rho\dot{u} + \nabla p = 0 \tag{2.11}$$

where

$$p = \frac{g}{2}\rho^2 + \frac{\varepsilon^2}{3}\rho^2\ddot{\rho}. \tag{2.12}$$

Introducing the potential function

$$W = \frac{g}{2}\rho^2 - \frac{\varepsilon^2}{6}\rho\dot{\rho}^2 \tag{2.13}$$

and substituting it into (2.3), one arrives at the Green-Naghdi model which is presented in the form (2.1) and (2.3) with the potential function (2.12).

The group analysis method was applied to one-dimensional equations (2.8) and (2.10) in Bagderina and Chupakhin (2005).

CHAPTER III

GROUP ANALYSIS METHOD

In this chapter, the group analysis method is discussed. An introduction to this method can be found in various textbooks (cf. Ovsiannikov 1978), (Olver, 1986), (Ibragimov, 1999), (Meleshko, 2005).

3.1 Lie Groups

Consider a set of invertible point transformations

$$\bar{z}^i = \varphi^i(z; a), \ a \in \Delta, \ z \in V, \tag{3.1}$$

where $i = 1, 2, ..., N$, a is a parameter, and Δ is a symmetric interval in R^1. The set V is an open set in R^N.

If $z = (x, u)$, then we use the notation $\varphi = (f, g)$. Here $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$ is the vector of the independent variables, and $u = (u^1, u^2, ..., u^m) \in \mathbb{R}^m$ is the vector of the dependent variables. The transformation of the independent variables x, and the dependent variables u has the form

$$\bar{x}_i = f^i(x, u; a), \ \bar{u}^j = g^j(x, u; a), \tag{3.2}$$

where $i = 1, 2, ..., n$, $j = 1, 2, ..., m$, $(x, u) \in V \subset R^n \times R^m$, and the set V is open in $R^n \times R^m$.

3.1.1 One-Parameter Lie-Group of Transformations

Definition 1. A set of transformations (3.1) is called a local one-parameter Lie group if it has the following properties

1. $\varphi(z;0) = z$ for all $z \in V$;

2. $\varphi(\varphi(z;a), b) = \varphi(z; a+b)$ for all $a, b, a+b \in \Delta, z \in V$;

3. If for $a \in \Delta$ one has $\varphi(z;a) = z$ for all $z \in V$, then $a = 0$;

4. $\varphi \in C^\infty(V, \Delta)$.

The Lie group of transformations (3.2) is called a one-parameter Lie group of point transformations. For a Lie group of point transformations, the functions f^i and g^j can be written by Taylor series expansion with respect to the group parameter a in a neighborhood of $a = 0$

$$\begin{aligned} \bar{x}_i &= x_i + a \left.\frac{\partial f^i}{\partial a}\right|_{a=0} + O(a^2), \\ \bar{u}^j &= u^j + a \left.\frac{\partial g^j}{\partial a}\right|_{a=0} + O(a^2). \end{aligned} \tag{3.3}$$

The transformations $\tilde{x_i} \approx x_i + a\xi^{x_i}(x,u)$ and $\tilde{u^j} \approx u^j + a\zeta^{u^j}(x,u)$ are called infinitesimal transformations of the Lie group of transformations (3.2), where

$$\xi^{x_i}(x,u) = \left.\frac{\partial f^i(x,u;a)}{\partial a}\right|_{a=0}, \zeta^{u^j}(x,u) = \left.\frac{\partial g^j(x,u;a)}{\partial a}\right|_{a=0}.$$

The components $\xi = (\xi^{x_1}, \xi^{x_2}, ..., \xi^{x_n})$, $\zeta = (\zeta^{u^1}, \zeta^{u^2}, ..., \zeta^{u^m})$ are called the infinitesimal representation of (3.2). This can be written in terms of the first-order differential operator

$$X = \xi^{x_i}(x,u)\partial_{x_i} + \zeta^{u^j}(x,u)\partial_{u^j}. \tag{3.4}$$

This operator X is called an infinitesimal generator.

There is a theorem, which relates a one-parameter Lie group G with its infinitesimal generator.

Theorem 1 (Lie). Let functions $f^i(x,u;a),\ i = 1, ..., n$ and $g^j(x,u;a),\ j = 1, ..., m$ satisfy the group properties and have the expansion

$$\bar{x}_i = f^i(x,u;a) \approx x_i + \xi^{x_i}(x,u)a,$$

$$\bar{u}^j = g^j(x,u;a) \approx u^j + \zeta^{u^j}(x,u)a$$

where

$$\xi^{x_i}(x,u) = \left.\frac{\partial f^i(x,u;a)}{\partial a}\right|_{a=0}, \zeta^{u^j}(x,u) = \left.\frac{\partial g^j(x,u;a)}{\partial a}\right|_{a=0}.$$

Then it solves the Cauchy problem

$$\frac{\mathrm{d}\bar{x}_i}{\mathrm{d}a} = \xi^{x_i}(\bar{x},\bar{u}), \ \frac{\mathrm{d}\bar{u}^j}{\mathrm{d}a} = \zeta^{u^j}(\bar{x},\bar{u}) \tag{3.5}$$

with the initial data

$$\bar{x}_i|_{a=0} = x_i, \ \bar{u}^j|_{a=0} = u^j. \tag{3.6}$$

Conversely, given $\xi^{x_i}(x,u)$ and $\zeta^{u^j}(x,u)$, the solution of the Cauchy problem (3.5), (3.6) forms a Lie group.

Equations (3.5) are called the Lie equations.

To apply a Lie group of transformations (3.2) for studying differential equations one needs to know how this group acts on the functions $u^j(x)$ and their derivatives. For the sake of simplicity, let us explain the basic idea for the case $n = 1$ and $m = 1$. Assume that $u_0(x)$ is a given known function, and the transformation is

$$\begin{aligned} \bar{x} &= f(x,u;a) \approx x + a\xi^x(x,u) \\ \bar{u} &= g(x,u;a) \approx u + a\zeta^u(x,u). \end{aligned} \tag{3.7}$$

Substituting $u_0(x)$ into the first equation (3.7), one obtains

$$\bar{x} = f(x, u_0(x); a).$$

Since $f(x, u_0(x); 0) = x$, the Jacobian at $a = 0$ is

$$\left.\frac{\partial \bar{x}}{\partial x}\right|_{a=0} = \left.\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial u}\frac{\partial u_0}{\partial x}\right)\right|_{a=0} = 1.$$

Thus, by virtue of the inverse function theorem, in some neighborhood of $a = 0$ one can express x as a function of $\bar{x}$ and a,

$$x = \theta(\bar{x}, a). \tag{3.8}$$

Note that after substituting (3.8) into the first equation (3.7), one has the identity

$$\bar{x} = f(\theta(\bar{x}, a), u_0(\theta(\bar{x}, a)); a). \tag{3.9}$$

Substituting (3.8) into the second equation (3.7), one obtains the transformed function

$$u_a(\bar{x}) = g(\theta(\bar{x}, a), u_0(\theta(\bar{x}, a)); a). \tag{3.10}$$

Differentiating equation (3.10) with respect to $\bar{x}$, one gets

$$\bar{u}_{\bar{x}} = \frac{\partial u_a(\bar{x})}{\partial \bar{x}} = \frac{\partial g}{\partial x}\frac{\partial \theta}{\partial \bar{x}} + \frac{\partial g}{\partial u}\frac{\partial u_0}{\partial x}\frac{\partial \theta}{\partial \bar{x}} = \left(\frac{\partial g}{\partial x} + \frac{\partial g}{\partial u}u_0'(x)\right)\frac{\partial \theta}{\partial \bar{x}},$$

where the derivative $\frac{\partial \theta}{\partial \bar{x}}$ can be found by differentiating equation (3.9) with respect to $\bar{x}$,

$$1 = \frac{\partial f}{\partial x}\frac{\partial \theta}{\partial \bar{x}} + \frac{\partial f}{\partial u}\frac{\partial u_0}{\partial x}\frac{\partial \theta}{\partial \bar{x}} = \left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial u}u_0'(x)\right)\frac{\partial \theta}{\partial \bar{x}}.$$

Since

$$\frac{\partial f}{\partial x}(\theta(\bar{x}, 0), u_0(\theta(\bar{x}, 0)); 0) = 1, \ \frac{\partial f}{\partial u}(\theta(\bar{x}, 0), u_0(\theta(\bar{x}, 0)); 0) = 0, \tag{3.11}$$

one has $\frac{\partial f}{\partial x} + \frac{\partial f}{\partial u}u_0'(x) \neq 0$ in some neighborhood of $a = 0$. Thus,

$$\frac{\partial \theta}{\partial \bar{x}} = \frac{1}{\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial u}u_0'(x)\right)},$$

and

$$\bar{u}_{\bar{x}} = \frac{\frac{\partial g(x,u_0;a)}{\partial x} + \frac{\partial g(x,u_0;a)}{\partial u}u_0'(x)}{\frac{\partial f(x,u_0;a)}{\partial x} + \frac{\partial f(x,u_0;a)}{\partial u}u_0'(x)} = h(x, u_0(x), u_0'(x); a).$$

Transformation (3.2) together with

$$\bar{u}_{\bar{x}} = h(x, u, u_x; a) \tag{3.12}$$

is called the prolongation of (3.2).

As before, the function h can be written by Taylor series expansion with respect to the parameter a in a neighborhood of the point $a = 0$:

$$\bar{u}_{\bar{x}} = h(x, u, u_x; a) \approx u_x + a\zeta^{u_x}(x, u, u_x), \tag{3.13}$$

where

$$\zeta^{u_x}(x,u,u_x) = \left.\frac{\partial h(x,u,u_x;a)}{\partial a}\right|_{a=0}, \quad h|_{a=0} = u_x.$$

Equation (3.12) can be rewritten as

$$h(x,u,u_x;a)\left(\frac{\partial f(x,u;a)}{\partial x} + u_x\frac{\partial f(x,u;a)}{\partial u}\right) = \left(\frac{\partial g(x,u;a)}{\partial x} + u_x\frac{\partial g(x,u;a)}{\partial u}\right).$$

Differentiating this equation with respect to the group parameter a and substituting $a = 0$, one finds

$$\left.\left(\frac{\partial h}{\partial a}\left(\frac{\partial f}{\partial x} + u_x\frac{\partial f}{\partial u}\right) + h\left(\frac{\partial^2 f}{\partial x\partial a} + u_x\frac{\partial^2 f}{\partial u\partial a}\right)\right)\right|_{a=0} = \left.\left(\frac{\partial^2 g}{\partial x\partial a} + u_x\frac{\partial^2 g}{\partial u\partial a}\right)\right|_{a=0}$$

or

$$\begin{aligned}
\zeta^{u_x}(x,u,u_x) &= \left.\frac{\partial h}{\partial a}\right|_{a=0}\left.\left(\frac{\partial f}{\partial x} + u_x\frac{\partial f}{\partial u}\right)\right|_{a=0} \\
&= \left.\left(\frac{\partial^2 g}{\partial x\partial a} + u_x\frac{\partial^2 g}{\partial u\partial a}\right)\right|_{a=0} - h|_{a=0}\left.\left(\frac{\partial^2 f}{\partial x\partial a} + u_x\frac{\partial^2 f}{\partial u\partial a}\right)\right|_{a=0} \\
&= \left(\frac{\partial \zeta^u}{\partial x} + u_x\frac{\partial \zeta^u}{\partial u}\right) - u_x\left(\frac{\partial \xi^x}{\partial x} + u_x\frac{\partial \xi^x}{\partial u}\right) \\
&= D_x(\zeta^u) - u_x D_x(\xi^x)
\end{aligned}$$

where

$$D_x = \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{xx}\frac{\partial}{\partial u_x} + \dots, \quad \xi^x = \left.\frac{\partial f}{\partial a}\right|_{a=0}, \quad \zeta^u = \left.\frac{\partial g}{\partial a}\right|_{a=0}, \quad \zeta^{u_x} = \left.\frac{\partial h}{\partial a}\right|_{a=0}.$$

The first prolongation of the generator (3.4) is given by

$$X^{(1)} = X + \zeta^{u_x}(x,u,u_x)\partial_{u_x}.$$

In the same way, one obtains the infinitesimal transformation of the second derivative

$$\bar{u}_{\bar{x}\bar{x}} \approx u_{xx} + a\zeta^{u_{xx}}(x,u,u_x,u_{xx}),$$

where $\zeta^{u_{xx}} = D_x(\zeta^{u_x}) - u_{xx}D_x(\xi^x)$, and the second prolongation of the generator (3.4) is

$$X^{(2)} = X^{(1)} + \zeta^{u_{xx}}(x,u,u_x,u_{xx})\partial_{u_{xx}}.$$

For constructing prolongations of an infinitesimal generator in case $n, m \geq 2$ one proceeds similarly.

Let $x = \{x_i\}$ be the set of independent variables and $u = \{u^j\}$ the set of dependent variables. The derivatives of the dependent variables are given by the sets $u_{(1)} = \left\{u_i^j\right\}$, $u_{(2)} = \left\{u_{is}^j\right\}, \ldots$, where $j = 1, \ldots, m$ and $i, s = 1, \ldots, n$. The derivatives of the differentiable functions u^j can be written in terms of the total differentiation operator D_i :

$$\begin{aligned} u_i^j &= D_i(u^j), \\ u_{is}^j &= D_s(u_i^j), \end{aligned}$$

where

$$D_i = \frac{\partial}{\partial x_i} + u_i^j \frac{\partial}{\partial u^j} + u_{is}^j \frac{\partial}{\partial u_s^j} + \ldots, \quad (i, s = 1, 2, \ldots, n;\ j = 1, 2, \ldots, m)\,. \quad (3.14)$$

The formula of the first prolongation of the generator $X = \xi^{x_i}(x, u)\partial_{x_i} + \zeta^{u^j}(x, u)\partial_{u^j}$ is

$$X^{(1)} = X + \zeta^{u_i^j}(x, u, u_{(1)})\partial_{u_i^j},$$

where

$$\zeta^{u_i^j} = D_i\left(\zeta^{u^j}\right) - u_s^j D_i\left(\xi^{x_s}\right) \quad ;\ i, s = 1, ..., n \ ;\ j = 1, ..., m.$$

The second prolongation of the generator X is

$$X^{(2)} = X^{(1)} + \zeta^{u_{i_1,i_2}^j}(x, u, u_{(1)}, u_{(2)})\partial_{u_{i_1,i_2}^j},$$

where

$$\zeta^{u_{i_1,i_2}^j} = D_{i_2}\left(\zeta^{u_{i_1}^j}\right) - u_{i_1,s}^j D_{i_2}\left(\xi^{x_s}\right) \quad ;\ i_1, i_2, s = 1, ..., n \ ;\ j = 1, ..., m. \quad (3.15)$$

In the general case, the k-th prolongation of the generator X is

$$X^{(k)} = X^{(k-1)} + \zeta^{u_{i_1,...,i_k}^j}(x, u, u_{(1)}, ..., u_{(k)})\partial_{u_{i_1,...,i_k}^j}$$

where

$$\zeta^{u^j_{i_1,\ldots,i_k}} = D_{i_k}\left(\zeta^{u^j_{i_1,\ldots,i_{k-1}}}\right) - u^j_{i_1,\ldots,i_{k-1},s} D_{i_k}\left(\xi^{x_s}\right);\ i_1,\ldots,i_k, s = 1,\ldots,n;\ j = 1,\ldots,m.$$

Lie groups of transformations are related with differential equations by the following.

Definition 2. Given a partial differential equation, a Lie group of transformations, which transforms a solution $u_0(x)$ into a solution $u_a(x)$ of the same equation is called an admitted Lie group of transformations.

Let $\mathcal{F} = (F^1, \ldots, F^k),\ k = 1, \ldots, N$ be differential functions of order p. The equations

$$F^k\left(x, u, u_{(1)}, u_{(2)}, \ldots, u_{(p)}\right) = 0,\ k = 1, \ldots, N \tag{3.16}$$

compose a manifold $[\mathcal{F} = 0]$ in the space of the variables $x, u, u_{(1)}, u_{(2)}, \ldots, u_{(p)}$.

After applying an admitted Lie group of transformations to a solution $u(x)$, one has

$$F^k\left(\bar{x}, \bar{u}, \bar{u}_{(1)}, \bar{u}_{(2)}, \ldots, \bar{u}_{(p)}\right) = 0,\ (k = 1, \ldots, N). \tag{3.17}$$

Differentiating these equations with respect to the group parameter a, and substituting $a = 0$, one finds

$$\left.\left(\frac{\partial F^k}{\partial x_i}\frac{\partial \bar{x}_i}{\partial a} + \frac{\partial F^k}{\partial u^j}\frac{\partial \bar{u}^j}{\partial a} + \frac{\partial F^k}{\partial u^j_{i_1}}\frac{\partial \bar{u}^j_{i_1}}{\partial a} + \ldots + \frac{\partial F^k}{\partial u^j_{i_1,i_2,\ldots,i_p}}\frac{\partial \bar{u}^j_{i_1,i_2,\ldots,i_p}}{\partial a}\right)\right|_{a=0} = 0$$

or

$$\xi^{x_i}\frac{\partial F^k}{\partial x_i} + \zeta^{u^j}\frac{\partial F^k}{\partial u^j} + \zeta^{u^j_{i_1}}\frac{\partial F^k}{\partial u^j_{i_1}} + \zeta^{u^j_{i_1,i_2}}\frac{\partial F^k}{\partial u^j_{i_1,i_2}} + \ldots + \zeta^{u^j_{i_1,i_2,\ldots,i_p}}\frac{\partial F^k}{\partial u^j_{i_1,i_2,\ldots,i_p}} = 0,$$

where

$$\xi^{x_i} = \left.\frac{\partial \bar{x}_i}{\partial a}\right|_{a=0},\quad \zeta^{u^j} = \left.\frac{\partial \bar{u}^j}{\partial a}\right|_{a=0},\quad \zeta^{u^j_{i_1}} = \left.\frac{\partial \bar{u}^j_{i_1}}{\partial a}\right|_{a=0},\ \ldots,\ \zeta^{u^j_{i_1,\ldots,i_p}} = \left.\frac{\partial \bar{u}^j_{i_1,\ldots,i_p}}{\partial a}\right|_{a=0}.$$

The last equation can be expressed as an action of the prolonged infinitesimal generator

$$X^{(p)}F^k\big|_{[\mathcal{F}=0]} = 0, \ (k = 1, ..., N), \tag{3.18}$$

where

$$X^{(p)} = \xi^{x_i}\frac{\partial}{\partial x_i} + \zeta^{u^j}\frac{\partial}{\partial u^j} + \zeta^{u^j_{i_1}}\frac{\partial}{\partial u^j_{i_1}} + \zeta^{u^j_{i_1,i_2}}\frac{\partial}{\partial u^j_{i_1,i_2}} + ... + \zeta^{u^j_{i_1,i_2,...,i_p}}\frac{\partial}{\partial u^j_{i_1,i_2,...,i_p}}.$$

Hence, in order to find the infinitesimal generator of the Lie group admitted by differential equations (3.16) one can use the following theorem.

Theorem 2. The differential equations (3.16) admits the group G with the generator X, if and only if, the following equations hold:

$$X^{(p)}F^k\big|_{[\mathcal{F}=0]} = 0, \ (k = 1, ..., N). \tag{3.19}$$

Equations (3.19) are called the determining equations.

3.1.2 Multi-Parameter Lie-Group of Transformations

Let O be a ball in the space R^r with a center at the origin. Assume that ψ is a mapping, $\psi : O \times O \longrightarrow R^r$. The pair (O, ψ) is called a local multi-parameter Lie group with the multiplication law ψ if it has the following properties:

1. $\psi(a, 0) = \psi(0, a) = a$ for all $a \in O$;
2. $\psi(\psi(a, b), c) = \psi(a, \psi(b, c))$ for all $a, b, c \in O$ for which $\psi(a, b), \psi(b, c) \in O$;
3. $\psi \in C^\infty(O, O)$.

Let V be an open set in R^N. Consider transformations

$$\bar{z}^i = \varphi^i(z; a), \tag{3.20}$$

where $i = 1, 2, ..., N$, $z \in V$, and $a \in O$ is a vector-parameter.

Definition 3. The set of transformations (3.20) is called a local r-parameter Lie group G^r if it has the following properties:

1. $\varphi(z,0) = z$ for all $z \in V$.
2. $\varphi(\varphi(z,a),b) = \varphi(z,\psi(a,b))$ for all $a, b, \psi(a,b) \in O,\ z \in V$.
3. If for $a \in O$ one has $\varphi(z,a) = z$ for all $z \in V$, then $a = 0$.

Note that if one fixes all parameters except one, for example a_k, then the multi-parameter Lie group of transformations (3.20) composes a one-parameter Lie group. Conversely, in group analysis it is proven that any r-parameter group is the union of one-parameter subgroups belonging to it.

Let G^r be a Lie group admitted by the system of partial differential equations

$$F^k(x,u,p) = 0,\ k = 1,...,s.$$

Assume that $\{X_1, X_2, ..., X_r\}$ is a basis of the Lie algebra L^r, which corresponds to the Lie group G^r.

Definition 4. A function $\Phi(x,u)$ is called invariant of a Lie group G^r if

$$\Phi(\bar{x},\bar{u}) = \Phi(x,u).$$

Theorem 3. A function $\Phi(x,u)$ is an invariant of the group G^r with the generators $X_i,\ (i = 1,...,r)$ if and only if,

$$X_i\Phi(x,u) = 0,\ (i = 1,...,r). \tag{3.21}$$

In order to find an invariant, one needs to solve the overdetermined system of linear equations (3.21). A set of functionally independent solutions of (3.21)

$$J = (\ J^1(x,u), J^2(x,u), ..., J^{m+n-r_*}(x,u)\ )$$

is called an universal invariant. Any invariant Φ can be expressed through this set

$$\Phi = \phi\ (\ J^1(x,u), J^2(x,u), ..., J^{m+n-r_*}(x,u)\ )\ .$$

Here n, m is the numbers of independent and dependent variables, respectively and r_* is the total rank of the matrix composed by the coefficients of the generators X_i, $(i = 1, 2, ..., r)$.

Definition 5. A set M is said to be invariant with respect to the group G^r, if the transformation (3.20) carries every point z of M to a point of M.

Definition 6. Let V be an open subset of R^N, and $\Psi : V \longrightarrow R^t$, $t \leq N$ a mapping belonging to the class $C^1(V)$. The system of equations $\Psi(z) = 0$ is called regular, if for any point $z \in V$:

$$rank \left(\frac{\partial(\psi^1, ..., \psi^t)}{\partial (z_1, ..., z_N)} \right) = t$$

where $\Psi = (\psi^1, ..., \psi^t)$.

If a system $\Psi(z) = 0$ is regular, then for each $z_0 \in V$ with $\Psi(z_0) = 0$ there exists a neighborhood U of z_0 in V such that

$$M = \{ z \in U \ : \ \Psi(z) = 0 \}$$

is a manifold. Such a manifold is called a regularly assigned manifold.

Theorem 4. A regularly assigned manifold M is an invariant manifold with respect to a Lie group G^r with the generator X_i, $(i = 1, ..., r)$, if

$$X_i \psi^k(z)\big|_M = 0, \ (i = 1, ..., r), \ k = 1, ..., t.$$

3.2 Lie algebra

Before giving the definition of a Lie algebra, one needs to introduce the commutator. Let $X_1 = \xi_1^i \partial_{x_i} + \zeta_1^j \partial_{u_j}$, $X_2 = \xi_2^i \partial_{x_i} + \zeta_2^j \partial_{u_j}$ be two generators. Let us define a new generator X, denoted by $[X_1, X_2]$, by the following formula

$$X = [X_1, X_2] = \left(X_1 \xi_2^i - X_2 \xi_1^i\right) \partial_{x_i} + \left(X_1 \zeta_2^j - X_2 \zeta_1^j\right) \partial_{u_j}.$$

The generator X is called the commutator of the generators X_1, X_2.

Definition 7. A vector space L over the field of real numbers with the operation of commutation $[\,\cdot\,,\,\cdot\,]$ is called a Lie algebra if $[X_1, X_2] \in L$ for any $X_1,\ X_2\ \in L$, and if the operation $[\,\cdot\,,\,\cdot\,]$ satisfies the axioms:

a.1 (bilinearity) : for any $X_1,\ X_2,\ X_3\ \in L$ and $a,\ b\ \in R$

$$\begin{aligned}[aX_1 + bX_2, X_3] &= a\,[X_1, X_3] + b\,[X_2, X_3] \\ [X_1, aX_2 + bX_3] &= a\,[X_1, X_2] + b\,[X_1, X_3]\end{aligned}$$

a.2 (antisymmetry) : for any $X_1,\ X_2\ \in L$

$$[X_1, X_2] = -\,[X_2, X_1]$$

a.3 (the Jacobi identity) : for any $X_1,\ X_2,\ X_3\ \in L$

$$[[X_1, X_2]\,, X_3] + [[X_2, X_3]\,, X_1] + [[X_3, X_1]\,, X_2] = 0.$$

Let L^r be an r-dimensional Lie algebra with basis $X_1, X_2, \ldots, X_r$: i.e., any vector $X \in L^r$ can be decomposed as

$$X = \sum_{k=1}^{r} x_k X_k$$

where x_k are the coordinates of the vector X in the basis $\{X_1, \ldots, X_r\}$. Then

$$[X_i, X_j] = \sum_{k=1}^{r} c_{ij}^{k} X_k; \quad i, j = 1, 2, \ldots, r$$

with real constants c_{ij}^k. The numbers c_{ij}^k are called the structural constants of the Lie algebra L^r for the basis $\{X_1, \ldots, X_r\}$.

Definition 8. A vector space $H \subset L$ is called a subalgebra of the Lie algebra L, if $[Y_1, Y_2] \in H$ for any $Y_1, Y_2\ \in H$.

Definition 9. A subalgebra $I \subset L$ is called an ideal of the Lie algebra L if for any $X \in L,\ Y \in I$ it is also true that $[X, Y] \in I$.

3.3 Classification of subalgebras

One of the main aims of group analysis is to construct exact solutions of differential equations. The set of all solutions can be divided into equivalence classes of solutions:

Definition 10. Two solutions u_1 and u_2 of a differential equation are said to be equivalent with respect to a Lie group G, if one of the solutions can be transformed into the other by a transformation belonging to the group G.

The problem of classification of exact solutions is equivalent to the classification of subgroups (or subalgebras) of the group G (or the subalgebra L). Because there is a one-to-one correspondence between Lie groups and Lie algebras let us explain here the classification of subalgebras. For this purpose, one needs the following definitions.

Definition 11. Let L and $\overline{L}$ be Lie algebras. A linear one-to-one map f of L onto $\overline{L}$ is called an isomorphism if it satisfies the equation

$$f\left([X_1, X_2]_L\right) = [f(X_1), f(X_2)]_{\overline{L}}, \ \forall \, X_1, \ X_2 \in L$$

where the indices L and $\overline{L}$ denote the commutators in the corresponding algebras. An isomorphism of L onto itself is called an automorphism of the Lie algebra L. This mapping will be denoted by the symbol $A : L \to L$.

In the finite-dimensional case, isomorphic Lie algebras have the same dimensions. The criterion for two Lie algebras to be isomorphic can be stated in terms of their structural constants. Two Lie algebras L and $\overline{L}$ are isomorphic, if and only if there exist bases for each of them in which their structural constants are equal.

Let L be a Lie algebra with basis $\{X_1, X_2, \ldots, X_n\}$. Then one has

$$[X_i, X_j] = \sum_{\alpha=1}^{n} c_{ij}^{\alpha} X_{\alpha}; \quad (i, j = 1, 2, \ldots, n),$$

where c_{ij}^{α} are the structural constants. One constructs a one-parameter family of automorphism, A_i, $(i = 1, ..., n)$ on L,

$$A_i : \sum_{i=1}^{n} x_i X_i \rightarrow \sum_{i=1}^{n} \bar{x}_i X_i$$

where $\bar{x}_i = \bar{x}_i(a)$, as follows. Consider the system

$$\frac{\mathrm{d}\bar{x}_j}{\mathrm{d}a} = \sum_{\beta=1}^{n} c_{\beta i}^{j} \bar{x}_{\beta}, \quad (j = 1, 2, \ldots, n). \tag{3.22}$$

Initial values for this system are $\bar{x}_j = x_j$ at $a = 0$. The set of solutions of these equations determines the set of automorphisms $\{A_i\}$.

The set of all subalgebras is divided into equivalence classes with respect to these automorphisms. A list of representatives, where each element of this list is one representative from every class, is called an optimal system of subalgebras.

Because of the difficulties in constructing the optimal system of subalgebras for Lie algebras of large dimension, there is a two-step algorithm (Ovsiannikov, 1994), which reduces this problem to the problem for constructing an optimal system of algebras of lower dimensions. In brief, let us consider an algebra L^r with basis $\{X_1, X_2, \ldots, X_r\}$. According to the algorithm, the algebra L^r is decomposed as $I_1 \oplus N_1$, where I_1 is an ideal of L^r and N_1 is a subalgebra of the algebra L^r. In the same way, the subalgebra N_1 can also be decomposed as $N_1 = I_2 \oplus N_2$. Repeat is the same process $(\alpha - 1)$ times one ends up with an algebra N_{α}, for which an optimal system of subalgebras can be easily constructed. By gluing the ideals I_l and subalgebras N_l starting from $l = \alpha$ to $l = 1$, together one constructs the optimal system of subalgebras for the algebra L^r. Note that for every subalgebra N_l one needs to check the subalgebra conditions and use the automorphisms to simplify the coefficients of these systems. Therefore, the problem for constructing an optimal system of subalgebras of the algebra L^r by this method is reduced to the problem of classification of algebras of lower dimensions.

After constructing the optimal system, one can start seeking invariant and partially invariant solutions of subalgebras from the optimal system.

3.4 Equivalence group of transformations

A system of PDES can be classified by the symbol $E(m,n,s,l)$, where m is the number of the dependent variables, n is the number of the independent variable, s is the order of the highest derivative and l is the number of differential equations. Normally the differential equations include arbitrary elements (θ). For searching Lie groups which are admitted by the original system, one needs to determine a group of transformations that changes arbitrary elements but does not change the differential structure. An infinitesimal approach (Meleshko, 1996) was applied for finding this group.

A nondegenerate change of dependent, independent variables and arbitrary elements which transfers any system of the differential equations of the given class

$$F_l(x,u,p,\theta) = 0 \tag{3.23}$$

to the system of the equations of the same class but with different arbitrary elements is called an equivalence transformation. Here p defines the partial derivatives $(u_{(1)}, u_{(2)}, ..., u_{(s)})$.

A Lie group of equivalence transformations with parameter a can be written as follows

$$\bar{x}_i = \phi^i(x,u,\theta;a),\ \ \bar{u}_j = \psi^j(x,u,\theta;a),\ \ \bar{\theta}_k = \Pi^k(x,u,\theta;a), \tag{3.24}$$

where $\theta_k = (\theta_{k^1}, \theta_{k^2}, ..., \theta_{k^\gamma})$ is the set of arbitrary elements. The generator of this group has the form

$$X^e = \xi^i \partial_{x_i} + \zeta^j \partial_{u_j} + \zeta^{\theta^k} \partial_{\theta_k},$$

where

$$\xi^i = \xi^i(x,u,\theta) = \frac{\partial \phi^i}{\partial a}|_{a=0},\ \zeta^j = \zeta^j(x,u,\theta) = \frac{\partial \psi^j}{\partial a}|_{a=0},\ \zeta^{\theta^k} = \zeta^{\theta^k}(x,u,\theta) = \frac{\partial \Pi^k}{\partial a}|_{a=0},$$

Transformations of arbitrary elements are obtained in the following way. Let $\theta_0(\theta, u)$ be given. By the inverse function theorem with equation (3.23), we can find $x = f(\bar{x}, \bar{u}; a)$ and $u = g(\bar{x}, \bar{u}; a)$. The transformed vector of arbitrary elements is

$$\theta_0(\bar{x}, \bar{u}) = \Pi(f(\bar{x}, \bar{u}; a), g(\bar{x}, \bar{u}; a), \theta_0(f(\bar{x}, \bar{u}; a), g(\bar{x}, \bar{u}; a)).$$

If $u_0(x)$ is solution of system (3.22) and $\theta_0(x, u)$ is a concrete value of the arbitrary elements, then we have

$$\bar{x} = \Phi(x, u_0(x), \theta_0(x, u_0(x)); a).$$

By the inverse function theorem, we can find

$$x = f(\bar{x}; a)$$

and we also obtain the transformed function

$$u_a(\bar{x}) = \Psi(f(\bar{x}, a), u_0(f(\bar{x}, a)), \theta_0(f(\bar{x}, a), u_0(f(\bar{x}, a)); a)). \tag{3.25}$$

Differentiating (3.24) with respect to $\bar{x}$, we get the transformation of derivatives $\bar{p}$. Since $u_a(\bar{x})$ is a solution of the same system with transformed arbitrary elements $\theta_a(\bar{x}, \bar{u})$ then

$$F_l(\bar{x}, u_a(\bar{x}), \bar{p_a}(\bar{x}), \theta_a(\bar{x}, u_a(\bar{x}))) = 0, \quad l = 1, 2, ...$$

The s-th prolongation of the infinitesimal generator X^e is

$$\bar{X}_e^{[s]} = X^e + \zeta_i^j \partial_{u_i^j} + \zeta_{x_i}^{\theta^k} \partial_{\theta_{x_i}^k} + \zeta_{u_j}^{\theta^k} \partial_{\theta_{u_j}^k} \tag{3.26}$$

where

$$\zeta_i^j = D_{x_i}\zeta^j - u_\alpha^j D_{x_i}\xi^\alpha,$$

$$\zeta_{x_i}^{\theta^k} = D_{x_i}^e \zeta^{\theta^k} - \theta_{x_\alpha}^k D_{x_i}^e \zeta^\alpha - \theta_{u^\beta}^k D_{x_i}^e \zeta^\beta,$$

$$\zeta_{u_j}^{\theta^k} = D_{x_i}^e \zeta^{\theta^k} - \theta_{x_\alpha}^k D_{u_j}^e \zeta^\alpha - \theta_{u^\beta}^k D_{u_j}^e \zeta^\beta.$$

Here

$$D_{x_i} = \frac{\partial}{\partial x_i} + u_i^j \frac{\partial}{\partial u_j} + (\theta_{x_i}^k + \theta_{u_j}^k u_i^j)\frac{\partial}{\partial \theta_k} + ...,$$

$$D_{x_i}^e = \frac{\partial}{\partial x_i} + \theta_{x_i}^k \frac{\partial}{\partial \theta^k} + ..., \quad D_{u^j}^e = \frac{\partial}{\partial u^j} + \theta_{u^j}^k \frac{\partial}{\partial \theta^k} + ...$$

By the same way as for the admitted Lie group, one can obtain the determining equations for the equivalence Lie group.

Let $G(\theta)$ be admitted by the equations for all arbitrary elements. The group $G(\theta)$ is called a kernel of groups. The corresponding Lie-algebra is called a kernel of Lie algebras.

CHAPTER IV

GROUP CLASSIFICATION OF THE ONE-DIMENSIONAL EQUATIONS

4.1 Introduction

Symmetry is a fundamental topic in many areas of physics and mathematics (Ovsiannikov[1978],Olver[1986],MarsdenRatiu[1994],GolubitskyStewart[2002]). Whereas group-theoretical methods play a prominent role in modern theoretical physics, a systematic use of them in constructing models of continuum mechanics has not been widely applied yet (Ovsiannikov[1994]). The present paper tries to help to fill this niche.

This manuscript is focused on group classification of a class of dispersive models (GavrilyukTeshukov2001)

$$\begin{gathered}\dot{\rho} + \rho\, div(u) = 0, \quad \rho\dot{u} + \nabla p = 0, \quad \dot{S} = 0, \\ p = \rho\frac{\delta W}{\delta\rho} - W = \rho\left(\frac{\partial W}{\partial\rho} - \frac{\partial}{\partial t}\left(\frac{\partial W}{\partial\dot{\rho}}\right) - div\left(\frac{\partial W}{\partial\dot{\rho}}u\right)\right) - W,\end{gathered} \tag{4.1}$$

where t is time, ∇ is the gradient operator with respect to space variables, ρ is the fluid density, u is the velocity field, $W(\rho, \dot{\rho}, S)$ is a given potential, "dot" denotes the material time derivative: $\dot{f} = \frac{df}{dt} = f_t + u\nabla f$ and $\frac{\delta W}{\delta\rho}$ denotes the variational derivative of W with respect to ρ at a fixed value of u. These models include the non-linear one-velocity model of a bubbly fluid (with incompressible liquid phase) at small volume concentration of gas bubbles (Iordanski (1960) , Kogarko (1961) ,Wijngaarden (1968) ,(Wijngaarden[1968]), and the dispersive shallow water model (Green & Naghdi (1975) , Salmon (1998). Equations (4.1) were obtained in (GavrilyukTeshukov2001) using the Lagrangian of the form

$$L = \frac{1}{2}|u|^2 - W(\rho, \dot{\rho}, S).$$

This is an example of a medium behavior dependent not only on thermodynamical variables but also on their derivatives with respect to space and time. In this particular case the potential function depends on the total derivative of the density which reflects the dependence of the medium on its inertia. Another example of models where the medium behavior depends on the derivatives is constructed in (GavrilyukShugrin[1996]) by assuming that the Lagrangian is of the form:

$$L = \frac{1}{2}|u|^2 - \varepsilon(\rho, |\nabla\rho|, S).$$

One of the methods for studying properties of differential equations is group analysis (Ovsiannikov[1978],Olver[1986],Ibragimov[1999]). This method is a basic method for constructing exact solutions of partial differential equations. A wide range of applications of group analysis to partial differential equations are collected in (HandbookLie(v1),bk:HandbookLie(v2),bk:HandbookLie(v3)). Group analysis, besides facilitating the construction of exact solutions, provides a regular procedure for mathematical modeling by classifying differential equations with respect to arbitrary elements. This feature of group analysis is the fundamental basis for mathematical modeling in the present paper.

An application of group analysis employs several steps. The first step is a group classification with respect to arbitrary elements. An algorithm of the group classification is applied in case where a system of differential equations has arbitrary elements in form of undefined parameters and functions. This algorithm is necessary since a specialization of the arbitrary elements can lead to an extension of admitted Lie groups. Group classification selects the functions $W(\rho, \dot{\rho}, S)$ such that the fluid dynamics equations (4.1) possess additional symmetry properties extending the kernel of admitted Lie groups. Algorithms of finding equivalence and admitted Lie groups are particular parts of the algorithm of the group classification.

A complete group classification of equations (4.1), where $W = W(\rho, \dot{\rho})$ is performed in (HematulinMeleshkoGavrilyuk[2007]) (one-dimensional case) and (SiriwatMeleshko[2008]) (three-dimensional case). Invariant solutions of some particular cases which are separated out by the group classification are considered in (HematulinMeleshkoGavrilyuk[2007],SiriwatMeleshko[2008],HematulinSiriwat[2009]). Group classification of the class of models describing the behavior of a dispersive continuum with $\varepsilon = \varepsilon(\rho, |\nabla\rho|)$ was studied in (VorakaMeleshko[2009]). It is also worth to notice that the classical gas dynamics model corresponds to $W = W(\rho, S)$ (or $\varepsilon = \varepsilon(\rho, S)$). A complete group classification of the gas dynamics equations was presented in (Ovsiannikov[1978]). Later, an exhausted program of studying the models appeared in the group classification of the gas dynamics equations was announced in (Ovsiannikov[1994]). Some results of this program were summarized in (Ovsiannikov[1999]).

The present paper is focused on the group classification of the one-dimensional equations of fluids (4.1), where the function $W = W(\rho, \dot{\rho}, S)$ satisfies the conditions $W_{S\dot{\rho}\dot{\rho}} = 0$ and $W_S \neq 0$.

The paper is organized as follows. The next section studies the equivalence Lie group of transformations. The equivalence transformations are applied for simplifying the function $W(\rho, \dot{\rho}, S)$ in the process of the classification. In Section 3 the defining equations of the admitted Lie group are presented. Analysis of these equations separates equations (4.1) into equivalent classes. Notice that these classes are defined by the function $W(\rho, \dot{\rho}, S)$. For convenience of the reader . The result of the group classification of equations (4.1) where $W_{S\dot{\rho}\dot{\rho}} = 0$ and $W_S \neq 0$ is summarized in Table 5.1. The admitted Lie algebras are also presented in this table.

4.2 Equivalence Lie group

For finding an equivalence Lie group the algorithm described in (Meleshko[1996],Meleshko[2005]) is applied. This algorithm differs from the classical one (Ovsiannikov[1978]) by assuming dependence of all coefficients from all variables including the arbitrary elements. Since the function W depends on the derivatives of the dependent variables and in order to simplify the process of finding an equivalence Lie group, new dependent variables are introduced:

$$u_3 = \dot{\rho}\,, u_4 = S.$$

Here $u_1 = \rho$, $u_2 = u$, $u_3 = \dot{\rho}$ and $u_4 = S$, $x_1 = x$, $x_2 = t$. An infinitesimal operator X^e of the equivalence Lie group is sought in the form (Meleshko[2005])

$$X^e = \xi^i \partial_{x_i} + \zeta^{u_j} \partial_{u_j} + \zeta^W \partial_W,$$

where all the coefficients ξ^i, ζ^{u_j}, $(i = 1, 2;\ j = 1, 2, 3, 4)$ and ζ^W are functions of the variables x, t, ρ, u, $\dot{\rho}$, S, W. Here after a sum over repeated indices is implied. The coefficients of the prolonged operator are obtained by using the prolongation formulae:

$$\zeta^{u_{\beta,i}} = D_i^e \zeta^{u_\beta} - u_{\beta,1} D_i^e \xi^x - u_{\beta,2} D_i^e \xi^t,\ (i = 1, 2),$$

$$D_1^e = \partial_x + u_{\beta,1}\partial_{u_\beta} + (\rho_x W_{\alpha,1} + \dot{\rho}_x W_{\alpha,2} + S_x W_{\alpha,3})\partial_{W_\alpha},$$
$$D_2^e = \partial_t + u_{\beta,2}\partial_{u_\beta} + (\rho_t W_{\alpha,1} + \dot{\rho}_t W_{\alpha,2} + S_t W_{\alpha,3})\partial_{W_\alpha},$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $\beta = (\beta_1, \beta_2)$ are multi-indexes ($\alpha_i \geq 0$), ($\beta_i \geq 0$)

$$(\alpha_1, \alpha_2, \alpha_3), 1 = (\alpha_1 + 1, \alpha_2, \alpha_3), \quad (\alpha_1, \alpha_2, \alpha_3), 2 = (\alpha_1, \alpha_2 + 1, \alpha_3),$$
$$(\alpha_1, \alpha_2, \alpha_3), 3 = (\alpha_1, \alpha_2, \alpha_3 + 1)$$

$$(\beta_1, \beta_2), 1 = (\beta_1 + 1, \beta_2), \quad (\beta_1, \beta_2), 2 = (\beta_1, \beta_2 + 1)$$

$$u_{(\beta_1,\beta_2)} = \frac{\partial^{\beta_1+\beta_2} u}{\partial x^{\beta_1} \partial t^{\beta_2}}, \quad W_{(\alpha_1,\alpha_2,\alpha_3)} = \frac{\partial^{\alpha_1+\alpha_2+\alpha_3} W}{\partial \rho^{\alpha_1} \partial \dot{\rho}^{\alpha_2} \partial S^{\alpha_3}}.$$

The conditions that W does not depend on t, x and u give

$$\zeta^{u_1}_{x_i} = 0, \ \zeta^{u_1}_{u} = 0, \ \zeta^{u_3}_{x_i} = 0, \ \zeta^{u_3}_{u} = 0, \ \zeta^{u_4}_{x_i} = 0, \ \zeta^{u_4}_{u} = 0, \ \zeta^{W}_{x_i} = 0, \ \zeta^{W}_{u_j} = 0, \ (i = 1, 2).$$

Using these relations, the prolongation formulae for the coefficients ζ^{W_α} become:

$$\zeta^{W_{\alpha,i}} = \widetilde{D}^e_i \zeta^{W_\alpha} - W_{\alpha,1} \widetilde{D}^e_i \zeta^{u_1} - W_{\alpha,2} \widetilde{D}^e_i \zeta^{u_3} - W_{\alpha,3} \widetilde{D}^e_i \zeta^{u_4}, \ (i = 1, 2),$$

$$\widetilde{D}^e_1 = \partial_\rho + W_{\alpha,1} \partial_{W_\alpha}, \ \widetilde{D}^e_2 = \partial_{\dot{\rho}} + W_{\alpha,2} \partial_{W_\alpha}, \ \widetilde{D}^e_3 = \partial_S + W_{\alpha,3} \partial_{W_\alpha}.$$

For constructing the determining equations and for their solution, the symbolic computer Reduce (Hearn) program was applied. Calculations give the following basis of generators of the equivalence Lie group

$$X^e_1 = \partial_x, \ X^e_2 = \partial_t, \ X^e_3 = t\partial_x + \partial_u, \ X^e_4 = t\partial_t + x\partial_x,$$
$$X^e_5 = t\partial_t + 2\rho\partial_\rho - u\partial_u, \ X^e_6 = \partial_W, \ X^e_7 = -u\partial_u + \rho\partial_\rho - W\partial_W + t\partial_t,$$
$$X^e_8 = \rho\varphi(S)\partial_W, \ X^e_9 = \dot{\rho} g(\rho, S)\partial_W, \ X^e_{10} = h(S)\partial_S,$$

where the functions $g(\rho, S), \varphi(S)$ and $h(S)$ are arbitrary. Here only the essential part of the operators X^e_i, $(i = 5, 6, ..., 10)$ is written.

Since the equivalence transformations corresponding to the operators X^e_5 , X^e_6, X^e_7, X^e_8,X^e_9 and X^e_{10} are applied for simplifying the function W in the process of the group classification, let us present these transformations. Because the function

W depends on ρ, $\dot{\rho}$ and S only, the transformations of these variables are presented:

$$
\begin{array}{lllll}
X_5^e: & \rho' = \rho e^{2a}, & \dot{\rho}' = \dot{\rho}, & S' = S & W' = W; \\
X_6^e: & \rho' = \rho, & \dot{\rho}' = \dot{\rho}, & S' = S & W' = We^{-2a}; \\
X_7^e: & \rho' = \rho e^{a}, & \dot{\rho}' = \dot{\rho}, & S' = S & W' = W + a; \\
X_8^e: & \rho' = \rho, & \dot{\rho}' = \dot{\rho}, & S' = S & W' = \rho\varphi(S)a + W; \\
X_9^e: & \rho' = \rho, & \dot{\rho}' = \dot{\rho}, & S' = S & W' = \dot{\rho}g(\rho, S)a + W \\
X_{10}^e: & \rho' = \rho, & \dot{\rho}' = \dot{\rho}, & S' = q(S,a) & W' = W;
\end{array}
$$

Here a is the group parameter.

4.3 Defining equations of the admitted Lie group

An admitted generator X is sought in the form

$$X = \xi^x\partial_x + \xi^t\partial_t + \zeta^\rho\partial_\rho + \zeta^u\partial_u + \zeta^S\partial_S,$$

where the coefficients $\xi^x, \xi^t, \zeta^\rho, \zeta^u, \zeta^S$ are functions of (x, t, ρ, u, S). Calculations showed that

$$\xi^x = (k_2t + k_3)x + k_4t + k_5, \quad \xi^t = k_2t^2 + (k_1 + 2k_3)t + k_7.$$

$$\zeta^\rho = \rho(k_8 - k_2t), \ \zeta^u = k_2(-ut + x) - u(k_1 + k_3) + k_4,$$

$$\zeta^S = \zeta^S(S),$$

$$k_2(3W_{\dot{\rho}\dot{\rho}\dot{\rho}}\dot{\rho} + W_{\dot{\rho}\dot{\rho}\rho}\rho + 3W_{\dot{\rho}\dot{\rho}}) = 0, \tag{4.2}$$

$$\begin{gathered}-3W_{\dot{\rho}\dot{\rho}S}\dot{\rho}\zeta^S + 3W_{\dot{\rho}\dot{\rho}\dot{\rho}}\dot{\rho}^2(k_1 + 2k_3 - k_8) - 3W_{\dot{\rho}\dot{\rho}\rho}\dot{\rho}\rho k_8 \\ +3W_{\dot{\rho}\dot{\rho}}\dot{\rho}(2k_3 - k_8) - \rho k_2(3W_{\dot{\rho}\dot{\rho}} + W_{\dot{\rho}\dot{\rho}\rho}\rho) = 0,\end{gathered} \tag{4.3}$$

$$k_2(W_{\rho\rho\rho\dot{\rho}}\dot{\rho}\rho - W_{\rho\rho\dot{\rho}}\dot{\rho} + 3W_{\rho\rho\dot{\rho}\dot{\rho}}\dot{\rho}^2 - W_{\rho\rho\rho}\rho + W_{\rho\rho}) = 0, \tag{4.4}$$

$$\begin{gathered}\zeta^S\rho(W_{\rho\rho S} - W_{\rho\rho\dot{\rho}S}\dot{\rho}) - W_{\rho\rho\rho\dot{\rho}}\dot{\rho}\rho^2k_8 - W_{\rho\rho\dot{\rho}}\dot{\rho}\rho(2k_1 + 2k_3 + k_8) \\ +W_{\rho\rho\dot{\rho}\dot{\rho}}\dot{\rho}^2\rho(k_1 + 2k_3 - k_8) + W_{\rho\rho\rho}\rho^2k_8 + W_{\rho\rho}\rho(2k_1 + 2k_3 + k_8) \\ +k_2\dot{\rho}(W_{\rho\rho\dot{\rho}\dot{\rho}}\rho^2 + 5W_{\dot{\rho}\dot{\rho}\rho}\rho + 3W_{\dot{\rho}\dot{\rho}}) = 0,\end{gathered} \tag{4.5}$$

$$k_2(W_{\rho\rho\dot\rho S}\dot\rho\rho^2-3W_{\rho\dot\rho S}\dot\rho\rho+3W_{\rho\dot\rho\dot\rho S}\dot\rho^2\rho-3W_{\dot\rho\dot\rho S}\dot\rho^2+3W_{\dot\rho S}\dot\rho-W_{\rho\rho S}\rho^2+3W_{\rho S}\rho-3W_S)=0, \tag{4.6}$$

$$\begin{gathered}-W_{\rho\rho\dot\rho S}\dot\rho\rho^2k_8-2W_{\rho\dot\rho S}\dot\rho\rho k_1-2W_{\rho\dot\rho S}\dot\rho\rho k_3+W_{\rho\dot\rho S}\dot\rho\rho k_8+W_{\rho\dot\rho\dot\rho S}\dot\rho^2\rho(2k_3+k_1-k_8)\\ +W_{\dot\rho\dot\rho S}\dot\rho^2(k_8-k_1-2k_3)+W_{\dot\rho S}\dot\rho(2k_1+2k_3-k_8)+W_{\rho\rho S}\rho^2k_8+W_{\rho S}\rho(2k_1+2k_3-k_8)\\ +W_S(k_8-2k_1-2k_3)+\dot\rho\rho k_2(W_{\rho\dot\rho\dot\rho S}\rho+3W_{\dot\rho\dot\rho S})+\zeta^S_S(-W_{\rho\dot\rho S}\dot\rho\rho+W_{\dot\rho S}\dot\rho+W_{\rho S}\rho-W_S)\\ +\zeta^S(-W_{\rho\dot\rho SS}\dot\rho\rho+W_{\dot\rho SS}\dot\rho+W_{\rho SS}\rho-W_{SS})=0,\end{gathered} \tag{4.7}$$

where k_i, $(i=1,2,\dots,8)$ are constant. The determining equations (4.2)–(4.7) define the kernel of admitted Lie algebras and its extensions. The kernel of admitted Lie algebras is determined for all functions $W(\rho,\dot\rho,S)$ and it consists of the generators

$$Y_4=\partial_x, Y_5=\partial_t, Y_6=t\partial_x+\partial_u.$$

Extensions of the kernel depend on the value of the function $W(\rho,\dot\rho,S)$. They can only be operators of the form

$$k_1X_1+k_2X_2+k_3X_3+k_8X_8+\zeta^S\partial_S,$$

where

$$\begin{gathered}X_1=t\partial_t-u\partial_u-\dot\rho\partial_{\dot\rho},\\ X_2=t(x\partial_x+t^2\partial_t+(x-ut)\partial_u-t\rho\partial_\rho-(\rho+3t\dot\rho)\partial_{\dot\rho},\\ X_3=x\partial_x-u\partial_u+2t\partial_t-2\dot\rho\partial_{\dot\rho}-\rho\partial_\rho-\dot\rho\partial_{\dot\rho},\\ X_8=\rho\partial_\rho+\dot\rho\partial_{\dot\rho}.\end{gathered}$$

Since the function $W(\rho,\dot\rho,S)$ depends on $\dot\rho$, the term with $\partial_{\dot\rho}$ is also presented in the generators.

Relations between the constants k_1, k_2, k_3, k_8 and $\zeta^S(S)$ depend on the function $W(\rho,\dot\rho,S)$.

4.4 Case $k_2 \neq 0$

If $k_2 \neq 0$, then equation (4.2) gives

$$3W_{\dot{\rho}\dot{\rho}\dot{\rho}}\dot{\rho} + W_{\rho\dot{\rho}\dot{\rho}}\rho + 3W_{\dot{\rho}\dot{\rho}} = 0.$$

The general solution of this equation is

$$W(\rho, \dot{\rho}, \tilde{S}) = \rho^3 g(z, S) + \varphi_0(\rho, S), \tag{4.8}$$

where $z = \dot{\rho}\rho^{-3}$. Substituting (4.8) into (4.4), one obtains

$$\rho\varphi_{0\rho\rho\rho} - \varphi_{0\rho\rho} = 0.$$

The general solution of this equation is

$$\varphi_0 = \rho^3\mu(S) + \rho I(S) + J(S), \tag{4.9}$$

where without loss of the generality by virtue of the equivalence transformation corresponding to the operator X_8^e, it can be assumed that $I(S) = 0$. Equation (4.6) gives that $J' = 0$. By virtue of the equivalence transformation corresponding to X_7^e, it can also be assumed that $J = 0$. Substituting the obtained W into (4.3) and splitting it with respect to ρ, one obtains $g_{zzz} = 0$ or $g = \varphi_2(S)z^2$, where $\varphi_2 \neq 0$. Notice that the linear part of the function φ_2 is also omitted because of the equivalence transformations corresponding to the generator X_9^e. The remaining part of equation (4.3) becomes

$$\varphi_2'\zeta^S - 2\varphi_2(k_3 + k_8) = 0. \tag{4.10}$$

If $\varphi_2' = 0$ or $\varphi_2 = q \neq 0$, then $k_3 = -k_8$ and equation (4.5) becomes

$$\mu'\zeta^S + 2k_1\mu = 0. \tag{4.11}$$

For $\mu' = 0$ the function W does not depend on S. Since this case has been studied in (HematulinMeleshkoGavrilyuk[2007]), it is excluded from further study in the present paper. Thus, one has to assume that $\mu' \neq 0$. From (4.11) one gets $\zeta^S = -2k_1\mu/\mu'$. Changing the entropy $\tilde{S} = \mu(S)$, one has

$$W(\rho, \dot{\rho}, \tilde{S}) = q\frac{\dot{\rho}^2}{\rho^3} + \rho^3\tilde{S},$$

and the extension of the kernel is given by the generators

$$X_1 - 2\tilde{S}\partial_{\tilde{S}}, \quad X_2, \quad X_3 - X_8.$$

In the final Table 1 this is model M_1, where the tilde sign is omitted.

If $\varphi_2' \neq 0$, then from (4.3) and (4.10), one obtains

$$\zeta^S = 2\frac{\varphi_2}{\varphi_2'}(k_3 + k_8),$$

$$\mu'\varphi_2(k_3 + k_8) + \varphi_2'\mu(k_1 + k_3 + k_8) = 0. \tag{4.12}$$

If $\mu \neq 0$ then, the last equation defines

$$k_1 = -(k_3 + k_8)(1 + \frac{\mu'\varphi_2}{\mu\varphi_2'}). \tag{4.13}$$

Differentiation (4.13) with respect to S gives

$$(k_3 + k_8)(\frac{\mu'\varphi_2}{\mu\varphi_2'})' = 0. \tag{4.14}$$

If $(\frac{\mu'\varphi_2}{\mu\varphi_2'})' = 0$ or $\mu = q_1\varphi_2^k$, then the general solution of equations (4.2) - (4.7) is

$$W(\rho, \dot{\rho}, \tilde{S}) = \frac{\dot{\rho}^2}{\rho^3}\tilde{S} + q_1\rho^3\tilde{S}^k,$$

where $\tilde{S} = \varphi_2(S)$. The extension of the kernel is given by the generators

$$X_2, \quad X_3 - X_8, \quad X_8 - (k+1)X_1 + 2\tilde{S}\partial_{\tilde{S}}.$$

In the final Table 1 this is model M_2.

If $(\frac{\mu'\varphi_2}{\mu\varphi_2'})' \neq 0$, then the general solution of equations (4.2) - (4.7) is

$$W(\rho, \dot{\rho}, \tilde{S}) = \frac{\dot{\rho}^2}{\rho^3}\tilde{S} + \rho^3\mu(\tilde{S}), \quad (\mu \neq q_1\tilde{S}^k),$$

and the extension of the kernel is given by the generators

$$X_2, \quad X_3 - X_8.$$

In the final Table 1 this is model M_3.

If $\mu = 0$, then

$$W(\rho, \dot{\rho}, \tilde{S}) = \frac{\dot{\rho}^2}{\rho^3}\tilde{S},$$

and the extension of the kernel is given by the generators

$$X_1, \quad X_2, \quad X_3 - X_8, \quad X_8 + 2\tilde{S}\partial_{\tilde{S}}.$$

In the final Table 1 this is model M_4.

Remark. The last two cases do not satisfy the restriction $W_{\dot{\rho}\dot{\rho}S} \neq 0$ announced in the title. For the case where $k_2 \neq 0$ it is not necessary to separate the study into the cases $W_{\dot{\rho}\dot{\rho}S} \neq 0$ and $W_{\dot{\rho}\dot{\rho}S} = 0$. Whereas for the analysis of the case where $k_2 = 0$, one needs to make this separation.

4.5 Results of the group classification

The result of the group classification of equations (4.1) is summarized in Table[1]. The linear part with respect to $\dot{\rho}$ of the function $W(\rho, \dot{\rho}, S)$ is omitted. The equivalence transformation corresponding to the operator X_{10}^e is also used. This transformation allows one to simplify the dependence on entropy of the function $W(\rho, \dot{\rho}, S)$.

The first column in Table[1] presents the number of the extension, forms of the function $W(\rho, \dot{\rho}, S)$ are presented in the second column, extensions of the kernel of admitted Lie algebras are given in the third column, restrictions for functions and constants are in the fourth column.

CHAPTER V

GROUP CLASSIFICATION OF THE ONE-DIMENSION NONISENTROPIC EQUATION

Case $k_2 = 0$

For further study the knowledge of $\zeta^S(S)$ plays a key role. For example, for $k_2 = 0$ equation (4.3) becomes

$$W_{\dot\rho\dot\rho S}\zeta^S = W_{\dot\rho\dot\rho\dot\rho}\dot\rho k_1 + 2k_3(W_{\dot\rho\dot\rho\dot\rho}\dot\rho + W_{\dot\rho\dot\rho}) - k_8(W_{\dot\rho\dot\rho\dot\rho}\dot\rho + W_{\dot\rho\dot\rho\rho}\rho + W_{\dot\rho\dot\rho}). \tag{5.1}$$

In the present paper we study the case where

$$W_{\dot\rho\dot\rho S} = 0.$$

By virtue of the equivalence transformation corresponding to the generator X_9^e, the general solution of the equation $W_{\dot\rho\dot\rho S} = 0$ is

$$W(\rho, \dot\rho, \tilde{S}) = \phi(\rho, \dot\rho) + h(\rho, S),$$

where $\phi_{\dot\rho} h_S \neq 0$. Since for $\phi_{\dot\rho\dot\rho} = 0$ equations (4.1) are equivalent to the gas dynamics equations, it is assumed that $\phi_{\dot\rho\dot\rho} \neq 0$. Equation (5.1) reduces to

$$k_1 a + k_3 b - k_8 c = 0, \tag{5.2}$$

where

$$a = \dot\rho\phi_{\dot\rho\dot\rho\dot\rho}, \quad b = 2(\dot\rho\phi_{\dot\rho\dot\rho\dot\rho} + \phi_{\dot\rho\dot\rho}), \quad c = -(\dot\rho\phi_{\dot\rho\dot\rho\dot\rho} + \rho\phi_{\dot\rho\dot\rho\rho} + \phi_{\dot\rho\dot\rho}).$$

In the further study the following strategy is used. Notice that equation (4.5) is linear with respect to ζ^S with the coefficient $h_{\rho\rho S}$, i.e.,

$$h_{\rho\rho S}\zeta^S = A$$

with some function $A = A(\rho, \dot{\rho}, S)$ which is independent of ζ^S. If $h_{\rho\rho S} = 0$, then due to equivalence transformations one can also assume that

$$h(\rho, S) = \eta(\rho) + \mu(S),$$

where $\mu' \neq 0$. In this case equation (4.7) leads to

$$\zeta^S = (-2k_1\mu - 2k_3\mu + k_8\mu + c_0)/\mu',$$

where c_0 is an arbitrary constant. The admitted generator takes the form

$$X = k_1(X_1 - 2\tilde{S}\partial_{\tilde{S}}) + k_3(X_3 - 2\tilde{S}\partial_{\tilde{S}}) + k_8(X_8 + \tilde{S}\partial_{\tilde{S}}) + c_0\partial_{\tilde{S}}, \qquad (5.3)$$

where $\tilde{S} = \mu(S)$. Remaining equations are (4.3) and (4.5). The relations between constants k_1, k_3 and k_8 depend on the functions $\eta(\rho)$ and $\phi(\rho, \dot{\rho})$. If $h_{\rho\rho S} \neq 0$, then the function ζ^S is defined by equation (4.5). In this case one needs to satisfy the system of equations (4.3), (4.7) and the condition that $\zeta^S = \zeta^S(S)$.

The analysis of the relations between the constants k_1, k_3 and k_8, follows to the algorithm developed for the gas dynamics equations (Ovsiannikov[1978]): the vector space $Span(V)$, where the set V consists of the vectors (a, b, c) with $\rho, \dot{\rho}$ and S are changed, is analyzed. This algorithm allows one to study all possible subalgebras without omission.

5.0.1 $dim(Span(V)) = 3$

If the function $W(\rho, \dot{\rho}, S)$ is such that $\dim(Span(V)) = 3$, then equation (5.2) is only satisfied for

$$k_1 = 0, \quad k_3 = 0, \quad k_8 = 0.$$

In this case equations (4.5) and (4.7) become

$$\zeta^S h_{\rho\rho S} = 0, \quad \zeta^S_S(\rho h_{\rho S} - h_S) + \zeta^S(\rho h_{\rho SS} - h_{SS}) = 0.$$

Since for $\zeta^S = 0$ there are no extensions of the kernel of admitted Lie algebras, one has to consider $\zeta^S \neq 0$. The general solution of the first equation, after using the equivalence transformation corresponding to the generator X_8^e, is

$$h = \mu(S),$$

where $\mu' \neq 0$. The general solution of the second equation is $\zeta^S = c/\mu'$. Hence

$$W(\rho, \dot{\rho}, \tilde{S}) = \phi(\rho, \dot{\rho}) + \tilde{S},$$

and the extension of the kernel is given by the generator

$$\partial_{\tilde{S}},$$

where $\tilde{S} = \mu(S)$. In the final Table 1 this is model M_5.

5.0.2 $\dim(Span(V)) = 2$

There exists a constant vector $(\alpha, \beta, \gamma) \neq 0$, which is orthogonal to the set V:

$$\alpha a + \beta b + \gamma c = 0. \tag{5.4}$$

This means that the function $\phi(\rho, \dot{\rho})$ satisfies the equation

$$(\alpha + 2\beta + \gamma)\dot{\rho}\phi_{\dot{\rho}\dot{\rho}\dot{\rho}} + \gamma\rho\phi_{\dot{\rho}\dot{\rho}\rho} = -(2\beta + \gamma)\phi_{\dot{\rho}\dot{\rho}}. \tag{5.5}$$

The characteristic system of this equation is

$$\frac{d\dot{\rho}}{(\alpha + 2\beta + \gamma)\dot{\rho}} = \frac{d\rho}{\gamma\rho} = \frac{d\phi_{\dot{\rho}\dot{\rho}}}{-(2\beta + \gamma)\phi_{\dot{\rho}\dot{\rho}}}. \tag{5.6}$$

Case $\gamma = 0$

Because $\phi_{\dot{\rho}\dot{\rho}} \neq 0$ and $(\alpha, \beta, \gamma) \neq 0$, one has that $\alpha + 2\beta \neq 0$. The general solution of equation (5.5) is

$$\phi_{\dot{\rho}\dot{\rho}} = \tilde{\varphi}(\rho)\dot{\rho}^k, \tag{5.7}$$

where $\tilde{\varphi}(\rho) \neq 0$ is an arbitrary function and $k = 2\beta/(\alpha + 2\beta)$. Since $\dim(Span(V)) = 0$ for $(\rho\tilde{\varphi}'/\tilde{\varphi})' = 0$, one has to assume that $(\rho\tilde{\varphi}'/\tilde{\varphi})' \neq 0$.

Substitution of (5.7) into (5.2) gives

$$k_8\tilde{\varphi}'\rho - \tilde{\varphi}(k(k_1 + 2k_3 - k_8) + 2k_3 - k_8) = 0. \tag{5.8}$$

The case $k_8 \neq 0$ leads to $(\rho\tilde{\varphi}'/\tilde{\varphi})' = 0$. Hence, $k_8 = 0$ and equation (5.8) becomes

$$k(k_1 + 2k_3) + 2k_3 = 0. \tag{5.9}$$

Let $k = 0$. Due to equation (5.8) one gets $k_3 = 0$. Integrating (5.7), one finds $\phi = \varphi(\rho)\dot{\rho}^2$. Equation (4.5) becomes

$$h_{\rho\rho S}\zeta^S + 2h_{\rho\rho}k_1 = 0. \tag{5.10}$$

Assume that $h_{\rho\rho} = 0$, this means that after using the equivalence transformation corresponding to the generator X_8^e, one has that $h = \mu(S)$, where $\mu' \neq 0$. Equation (4.7) after integration gives

$$\zeta^S = -2k_1\mu/\mu' + c_0/\mu',$$

where c_0 is a constant of the integration. Thus,

$$W(\rho, \dot{\rho}, \tilde{S}) = \varphi(\rho)\dot{\rho}^2 + \tilde{S}.$$

and the extension of the kernel is given by the generators

$$\partial_{\tilde{S}}, \quad X_1 - 2\tilde{S}\partial_{\tilde{S}},$$

where $\tilde{S} = \mu(S)$. In the final Table 1 this is model M_6.

Assume that $h_{\rho\rho} \neq 0$. For the existence of an extension of the kernel, equation (5.10) implies that $h(\rho, S) = \eta(\rho)\mu(S) + \mu_2(S)$, where $\mu\eta'' \neq 0$. In this case equation (4.5) becomes

$$\mu'\zeta^S + 2k_1\mu = 0.$$

If $\mu' = 0$, then $\mu_2' \neq 0$, $k_1 = 0$ and equation (4.7) gives $\zeta^S = c_0/\mu_2'$. Thus,

$$W(\rho, \dot{\rho}, \tilde{S}) = \varphi(\rho)\dot{\rho}^2 + \eta(\rho) + \tilde{S},$$

and the extension of the kernel is given by the generator

$$\partial_{\tilde{S}},$$

where $\tilde{S} = \mu_2(S)$. In the final Table 1 this is model M_7.

If $\mu' \neq 0$, then $\zeta^S = -2k_1\mu/\mu'$, and equation (4.7) gives

$$(\mu_2'/\mu')' = 0.$$

Hence, without loss of generality one can assume that $\mu_2 = 0$. Therefore,

$$W(\rho, \dot{\rho}, \tilde{S}) = \varphi(\rho)\dot{\rho}^2 + \eta(\rho)\tilde{S}.$$

and the extension of the kernel is given by the generator

$$X_1 - 2\tilde{S}\partial_{\tilde{S}},$$

where $\tilde{S} = \mu(S)$. In the final Table 1 this is model M_8.

Remark. In the cases where $\mu' \neq 0$ one can assume that $\mu_2(S) = f(\mu(S))$. This simplifies calculations.

Let $k \neq 0$. Equation (5.9) gives

$$k_1 = -2k_3\frac{1+k}{k}.$$

The function $\phi(\rho, \dot{\rho})$ is obtained by integrating equation (5.7). The integration depends on the value of k.

Let $k = -1$, then

$$\phi = \varphi(\rho)\dot{\rho}\ln|\dot{\rho}|. \tag{5.11}$$

Substituting (5.11) into (4.5), one obtains

$$\zeta^S h_{\rho\rho S} + 2k_3 h_{\rho\rho} = 0. \tag{5.12}$$

If $h_{\rho\rho} = 0$, then $h = \mu(S)$ with $\mu' \neq 0$, and equation (4.7) leads to

$$\zeta^S = -2k_3\mu/\mu' + c_0/\mu'.$$

Therefore,

$$W(\rho, \dot{\rho}, \tilde{S}) = \varphi(\rho)\dot{\rho}\ln|\dot{\rho}| + \tilde{S},$$

and the extension of the kernel is given by the generators

$$X_3 - 2\tilde{S}\partial_{\tilde{S}}, \quad \partial_{\tilde{S}},$$

where $\tilde{S} = \mu(S)$. In the final Table 1 this is model M_9.

If $h_{\rho\rho} \neq 0$, then

$$h(\rho, S) = \mu(S)\eta(\rho) + \mu_2(S), \quad (\mu\eta'' \neq 0).$$

Equation (4.5) becomes $\mu'\zeta^S + 2k_3\mu = 0$.

If $\mu' = 0$, then $\mu_2' \neq 0$, $k_3 = 0$ and equation (4.7) gives $\zeta^S = c_0/\mu_2'$. Thus,

$$W(\rho, \dot{\rho}, \tilde{S}) = \varphi(\rho)\dot{\rho}\ln|\dot{\rho}| + \eta(\rho) + \tilde{S},$$

and the extension of the kernel is defined by the generator

$$\partial_{\tilde{S}},$$

where $\tilde{S} = \mu_2(S)$. In the final Table 1 this is model M_{10}.

If $\mu' \neq 0$, then

$$\zeta^S = -2k_3\mu(S)/\mu'.$$

Similar to the case $k = 0$, equation (4.7) gives $\mu_2 = 0$. Therefore

$$W(\rho, \dot{\rho}, \tilde{S}) = \varphi(\rho)\dot{\rho}\ln|\dot{\rho}| + \eta(\rho)\tilde{S}, \quad (\eta'' \neq 0),$$

and the extension of the kernel is given by the generator

$$X_3 - 2\tilde{S}\partial_{\tilde{S}},$$

where $\tilde{S} = \mu(S)$. In the final Table 1 this is model M_{11}.

Let $k = -2$, then

$$\phi = \varphi(\rho)\ln|\dot{\rho}|. \qquad (5.13)$$

Equation (4.5) becomes

$$\zeta^S h_{\rho\rho S} - k_3\varphi'' = 0 \qquad (5.14)$$

Assuming that $h_{\rho\rho S} = 0$, one has

$$h(\rho, S) = \eta(\rho) + \mu(S),$$

where $\mu' \neq 0$. Equation (4.7) leads to

$$\zeta^S = c_0/\mu'. \qquad (5.15)$$

Therefore

$$W(\rho, \dot{\rho}, \tilde{S}) = \varphi(\rho)\ln|\dot{\rho}| + \eta(\rho) + \tilde{S},$$

and (a) for $\varphi'' = 0$, one has two admitted generators

$$X_3 - X_1, \ \partial_{\tilde{S}},$$

(b) for $\varphi'' \neq 0$, there is the only admitted generator

$$\partial_{\tilde{S}}.$$

Here $\tilde{S} = \mu(S)$. In the final Table 1 case (a) is model M_{12} and case (b) is model M_{13}.

Assuming that $h_{\rho\rho S} \neq 0$, one has

$$\zeta^S = k_3 \frac{\varphi''}{h_{\rho\rho S}}.$$

Notice that here $k_3 \neq 0$, otherwise there is no an extension of the kernel of admitted Lie algebras. Hence,

$$\left(\frac{\varphi''}{h_{\rho\rho S}}\right)_\rho = 0. \tag{5.16}$$

If $\varphi'' = 0$, then equation (4.7) is also satisfied. Therefore there is the only extension

$$X_3 - X_1,$$

and

$$W(\rho, \dot{\rho}, \tilde{S}) = (q_1\rho + q_0)\ln|\dot{\rho}| + h(\rho, S),$$

where $(q_0^2 + q_1^2)h_{\rho\rho S} \neq 0$. In the final Table 1 this is model M_{14}.

If $\varphi'' \neq 0$, then equations (5.16) and (4.7) give

$$h(\rho, S) = \varphi(\rho)\mu(S) + \eta(\rho) + q_2\mu(S),$$

where $\mu' \neq 0$. Therefore,

$$W(\rho, \dot{\rho}, \tilde{S}) = \varphi(\rho)(\ln|\dot{\rho}| + \tilde{S}) + \eta(\rho) + q_2\tilde{S}, \ \ (\varphi'' \neq 0),$$

and the extension of the kernel is

$$X_3 - X_1 + \partial_{\tilde{S}},$$

where $\tilde{S} = \mu(S)$. In the final Table 1 this is model M_{15}.

Let $k(k+1)(k+2) \neq 0$ in (5.7), then

$$\phi = \varphi(\rho)\dot{\rho}^{(k+2)} \tag{5.17}$$

Substituting (5.17) into (4.5), one obtains

$$\zeta^S h_{S\rho\rho} k - 2k_3(k+2)h_{\rho\rho} = 0. \tag{5.18}$$

If $h_{\rho\rho} = 0$, then one can consider that $h = \mu(S)$, where $\mu' \neq 0$. Equation (4.7) is

$$\zeta^S = 2k_3 \frac{(k+2)}{k} \mu/\mu' + c_0/\mu'.$$

In this case

$$W(\rho, \dot{\rho}, \tilde{S}) = \dot{\rho}^{k+2}\varphi(\rho) + \tilde{S},$$

and the extension of the kernel is given by the generators

$$kX_3 - 2(k+1)X_1 + 2(k+2)\tilde{S}\partial_{\tilde{S}}, \quad \partial_{\tilde{S}},$$

where $\tilde{S} = \mu(S)$. In the final Table 1 this is model M_{16}.

If $h_{\rho\rho} \neq 0$, then for an existence of an extension of the kernel, equation (5.18) requires that

$$h(\rho, S) = \eta(\rho)\mu(S) + \mu_2(S),$$

where $\mu\eta'' \neq 0$. Equation (5.18) becomes

$$\zeta^S \mu' k - 2k_3(k+2)\mu = 0.$$

If $\mu' = 0$, then $\mu_2' \neq 0$, $k_3 = 0$ and equation (4.7) gives $\zeta^S = c_0/\mu_2'$. Thus,

$$W(\rho, \dot{\rho}, \tilde{S}) = \dot{\rho}^{k+2}\varphi(\rho) + \eta(\rho) + \tilde{S}, \quad (\eta'' \neq 0).$$

and the extension of the kernel is given by the generator

$$\partial_{\tilde{S}},$$

where $\tilde{S} = \mu_2(S)$. In the final Table 1 this is model M_{17}.

If $\mu' \neq 0$, then

$$\zeta^S = 2k_3 \frac{(k+2)}{k} \mu/\mu',$$

Similar to the case $k = 0$, equation (4.7) gives $\mu_2 = 0$. Therefore,

$$W(\rho, \dot{\rho}, \tilde{S}) = \dot{\rho}^{k+2}\varphi(\rho) + \eta(\rho)\tilde{S}, \quad (\eta'' \neq 0),$$

and the extension of the kernel is given by the generator

$$kX_3 - 2(k+1)X_1 + 2(k+2)\tilde{S}\partial_{\tilde{S}},$$

where $\tilde{S} = \mu(S)$. In the final Table 1 this is model M_{18}.

Case $\gamma \neq 0$.

In this case the general solution of (5.6) is

$$\phi = \rho^{\lambda} g(z), \quad (g'' \neq 0), \tag{5.19}$$

where $z = \dot{\rho}\rho^k$, $k = -(\alpha + 2\beta)/\gamma - 1$, $\lambda = 2(\beta + \alpha)/\gamma + 1$. Substituting ϕ into (5.1), one obtains

$$zg'''k_0 + g''\tilde{k_0} = 0, \tag{5.20}$$

where $k_0 = k_1 + 2k_3 - k_8(k+1)$ and $\tilde{k_0} = 2k_3 - k_8(2k + \lambda + 1)$. If $k_0 \neq 0$, then $\dim(Span(V)) \leq 1$, hence, $k_0 = 0$ and $\tilde{k_0} = 0$, which mean that

$$k_1 = -k_8(k + \lambda), \quad k_3 = k_8(2k + \lambda + 1)/2.$$

Equation (4.5) becomes

$$\zeta^S h_{S\rho\rho} + k_8(\rho h_{\rho\rho\rho} - (\lambda - 2)h_{\rho\rho}) = 0. \tag{5.21}$$

Assume that $h_{\rho\rho S} = 0$ or

$$h(\rho, S) = \eta(\rho) + \mu(S),$$

where $\mu' \neq 0$. Equation (4.5) and (4.7) become, respectively,

$$k_8(\rho\eta''' - (\lambda - 2)\eta'') = 0, \quad \zeta^S = k_8\lambda\mu/\mu' + c_0/\mu'.$$

If $\rho\eta''' - (\lambda - 2)\eta'' \neq 0$, then $k_8 = 0$. Thus,

$$W(\rho, \dot{\rho}, \tilde{S}) = \rho^{\lambda} g(\dot{\rho}\rho^{k}) + \eta(\rho) + \tilde{S},$$

and there is the only extension of the kernel of admitted Lie algebras corresponding to the generator

$$\partial_{\tilde{S}},$$

where $\widetilde{S} = \mu(S)$. In the final Table 1 this is model M_{19}.

If $\rho\eta''' - (\lambda - 2)\eta'' = 0$ or

$$\eta = \begin{cases} q_1\rho^2, & \lambda = 0, \\ q_1\rho\ln(\rho), & \lambda = 1, \\ q_1\rho^{\lambda}, & \lambda(\lambda - 1) \neq 0. \end{cases}$$

Then,

$$W(\rho, \dot{\rho}, \tilde{S}) = \rho^{\lambda} g(\dot{\rho}\rho^{k}) + \eta(\rho) + \tilde{S},$$

and the extension of the kernel of admitted Lie algebras corresponding to the generators is

$$-(k + \lambda)X_1 + \frac{(2k + \lambda + 1)}{2}X_3 + X_8 + \lambda\widetilde{S}\partial_{\widetilde{S}}, \ \ \partial_{\widetilde{S}},$$

where $\widetilde{S} = \mu(S)$. In the final Table 1 these models correspond to M_{20}-M_{22}.

Assume that $h_{S\rho\rho} \neq 0$ in (5.21), then

$$\zeta^S = -k_8(\rho h_{\rho\rho\rho} - (\lambda - 2)h_{\rho\rho})/h_{S\rho\rho}.$$

Since $\zeta^S = \zeta^S(S)$, one has

$$\frac{-\rho h_{\rho\rho\rho} + (\lambda - 2)h_{\rho\rho}}{h_{S\rho\rho}} = H(S), \tag{5.22}$$

and $\zeta^S = k_8 H(S)$.

If $H = 0$, then the general solution of (5.22) is

$$h_{\rho\rho}(\rho, S) = \mu(S)\rho^{\lambda - 2}. \tag{5.23}$$

Hence,

$$h(\rho, S) = \mu(S)\eta(\rho) + \mu_2(S),$$

where $\mu'(S) \neq 0$ and

$$\eta = \begin{cases} \ln \rho, & \lambda = 0, \\ \rho \ln(\rho), & \lambda = 1, \\ \rho^\lambda, & \lambda(\lambda - 1) \neq 0. \end{cases}$$

Equation (4.7) gives

$$k_8 \left(\lambda \mu_2' + \mu' \left(\rho^2 \eta'' - \lambda(\rho\eta' - \eta)\right)\right) = 0.$$

This equation leads to: (a) if $\lambda = 0$, then $k_8 = 0$, (b) if $\lambda \neq 0$, then $\mu_2' k_8 = 0$. Hence, an extension of the kernel of admitted Lie algebras occurs for $\lambda \neq 0$. In this case $\mu_2' = 0$, which allows one to assume that $\mu_2 = 0$. Thus,

$$W(\rho, \dot{\rho}, \tilde{S}) = \rho^\lambda g(\dot{\rho}\rho^k) + \tilde{S}\eta(\rho),$$

and the extension is given by the generator

$$-(k + \lambda)X_1 + \frac{(2k + \lambda + 1)}{2}X_3 + X_8.$$

In the final Table 1 these models correspond to M_{23}-M_{25}.

If $H \neq 0$, then equation (5.22) leads to

$$h = \rho^\lambda Q + \mu_2,$$

where $\mu = \mu(S)$, $\mu_2 = f(\mu(S))$, $Q = Q(z)$, $z = \rho\mu$ and $\mu' \neq 0$. Here $H(S) = \mu/\mu' \neq 0$. Substitution of

$$W(\rho, \dot{\rho}, \tilde{S}) = \rho^\lambda g(\dot{\rho}\rho^k) + \rho^{-\lambda} Q(\rho\mu(S)) + f(\mu(S))$$

into (4.7) gives

$$\mu f'' + (\lambda + 1) f' = 0.$$

Hence,

$$f' = c\mu^{-(\lambda+1)}.$$

Integration of this equation depends on λ:

$$\mu_2 = \begin{cases} q_1 \ln \mu, & \lambda = 0, \\ q_1 \mu^{-\lambda}, & \lambda \neq 0. \end{cases}$$

Thus,

$$\begin{aligned} \lambda = 0 &: \quad W(\rho, \dot{\rho}, \tilde{S}) = g(\dot{\rho}\rho^k) + Q(\rho\tilde{S}) + q_1 \ln \tilde{S}, \\ \lambda \neq 0 &: \quad W(\rho, \dot{\rho}, S) = \rho^\lambda(g(\dot{\rho}\rho^k) + Q(\rho\tilde{S})). \end{aligned}$$

The extension of the kernel is given by the generator

$$-(k+\lambda)X_1 + \frac{(2k+\lambda+1)}{2}X_3 + X_8 - \tilde{S}\partial_{\tilde{S}},$$

where $\tilde{S} = \mu(S)$. In the final Table 1 these models correspond to M_{26}-M_{27}.

5.0.3 $\dim(Span(V)) = 1$

Let $\dim(Span(V)) = 1$. There exists a constant vector $(\alpha, \beta, \gamma) \neq 0$ such that

$$(a, b, c) = (\alpha, \beta, \gamma)B$$

with some function $B(\rho, \dot{\rho}, S) \neq 0$. Since $\phi_{\dot{\rho}\dot{\rho}} \neq 0$, one has $\beta - 2\alpha \neq 0$, and

$$\rho\phi_{\rho\dot{\rho}\dot{\rho}} = \lambda\phi_{\dot{\rho}\dot{\rho}}, \quad \dot{\rho}\phi_{\dot{\rho}\dot{\rho}\dot{\rho}} = k\phi_{\dot{\rho}\dot{\rho}},$$

where

$$\lambda = \frac{3\alpha - \beta - \gamma}{\beta - 2\alpha}, \quad k = \frac{\alpha}{\beta - 2\alpha}.$$

These relations give

$$\phi_{\dot{\rho}\dot{\rho}} = c_1\rho^\lambda\dot{\rho}^k, \tag{5.24}$$

where $c_1 \neq 0$ is constant. Equation (4.3) becomes

$$k_1 k + 2k_3(k+1) - k_8(k+\lambda+1) = 0. \tag{5.25}$$

Integration of (5.24) depends on the value of k. Notice that $k^2+\lambda^2 \neq 0$, otherwise $\dim(Span(V)) = 0$.

Case $k=-1$.

Integrating (5.24), one obtains

$$\phi = q_0\rho^{\lambda}\dot{\rho}\ln|\dot{\rho}|. \tag{5.26}$$

Equation (5.25) gives

$$k_1 = -\lambda k_8,$$

and equation (4.5) becomes

$$h_{S\rho\rho}\zeta^S = -2k_3h_{\rho\rho} - k_8(\rho h_{\rho\rho\rho} - h_{\rho\rho}(2\lambda-1)). \tag{5.27}$$

Assuming that $h_{S\rho\rho} = 0$ or

$$h(\rho,S) = \eta(\rho) + \mu(S), \quad (\mu' \neq 0),$$

equation (5.27) is reduced to the equation

$$\rho\eta''' k_8 - (k_8(2\lambda-1) - 2k_3)\eta'' = 0. \tag{5.28}$$

The general solution of equation (4.7) is

$$\zeta^S = (k_8(2\lambda+1) - 2k_3)\frac{\mu}{\mu'} + \frac{c_0}{\mu'},$$

where c_0 is an arbitrary constant.

If $\eta'' = 0$, then without loss of the generality one can assume that $\eta = 0$. Equation (5.28) is satisfied. Thus,

$$W(\rho,\dot{\rho},\tilde{S}) = q_0\dot{\rho}\rho^{\lambda}\ln|\dot{\rho}| + \tilde{S},$$

and the extension of the kernel of admitted Lie algebras is defined by the generators

$$X_3 - 2\tilde{S}\partial_{\tilde{S}}, \quad X_8 - \lambda X_1 + (2\lambda+1)\tilde{S}\partial_{\tilde{S}}, \quad \partial_{\tilde{S}},$$

where $\tilde{S} = \mu(S)$. In the final Table 1 this model corresponds to M_{28}.

If $\eta'' \neq 0$, then

$$k_3 = k_8\left(\lambda - \frac{1}{2} - \frac{\rho\eta'''}{2\eta''}\right). \tag{5.29}$$

Because k_3 is constant, one has

$$k_8\left(\frac{\rho\eta'''}{\eta''}\right)' = 0.$$

Assume that

$$\left(\frac{\rho\eta'''}{\eta''}\right)' = 0$$

or $\eta'' = q_1\rho^\nu$, where ν is constant. Substituting η'' into (5.29), one gets

$$k_3 = k_8\left(\lambda - \frac{\nu+1}{2}\right).$$

Thus

$$W(\rho, \dot{\rho}, \tilde{S}) = q_0\dot{\rho}\rho^\lambda \ln|\dot{\rho}| + \eta(\rho) + \tilde{S}, \quad (\eta'' = q_1\rho^\nu, \quad q_1 \neq 0),$$

and the extension of the kernel of admitted Lie algebras is defined by the generators

$$2X_8 - 2\lambda X_1 + (2\lambda - \nu - 1)X_3 + 2(\nu+2)\tilde{S}\partial_{\tilde{S}}, \quad \partial_{\tilde{S}},$$

where $\tilde{S} = \mu(S)$ and $q_1 \neq 0$. In the final Table 1 this model corresponds to M_{29}.

If

$$\left(\frac{\rho\eta'''}{\eta''}\right)' \neq 0,$$

then $k_8 = 0$,

$$W(\rho, \dot{\rho}, \tilde{S}) = q_0\dot{\rho}\rho^\lambda \ln|\dot{\rho}| + \eta(\rho) + \tilde{S},$$

and the extension of the kernel of admitted Lie algebras is defined by the only generator

$$\partial_{\tilde{S}}.$$

In the final Table 1 this model corresponds to M_{30}.

Assuming that $h_{S\rho\rho} \neq 0$, equation (5.27) gives

$$\zeta^S = -2k_3 \frac{h_{\rho\rho}}{h_{S\rho\rho}} - k_8 \frac{\rho h_{\rho\rho\rho} - h_{\rho\rho}(2\lambda - 1)}{h_{S\rho\rho}}. \tag{5.30}$$

Differentiating equation (5.30) with respect to ρ, one obtains

$$2k_3 \left(\frac{h_{\rho\rho}}{h_{S\rho\rho}}\right)_\rho + k_8 \left(\frac{\rho h_{\rho\rho\rho} - h_{\rho\rho}(2\lambda - 1)}{h_{S\rho\rho}}\right)_\rho = 0. \tag{5.31}$$

If $\left(\frac{h_{\rho\rho}}{h_{S\rho\rho}}\right)_\rho = 0$, then $h = \eta(\rho)\mu(S) + f(\mu(S))$, and equation (5.31) becomes

$$k_8 \left(\frac{\rho\eta'''}{\eta''}\right)' = 0. \tag{5.32}$$

Here $\mu'\eta'' \neq 0$.

If $\left(\frac{\rho\eta'''}{\eta''}\right)' \neq 0$, then $k_8 = 0$. Equation (4.7) gives

$$f(\mu) = c_0\mu. \tag{5.33}$$

Changing the function η such that $\eta + c_0 \to \eta$, one obtains

$$W(\rho, \dot{\rho}, \tilde{S}) = q_0\dot{\rho}\rho^\lambda \ln|\dot{\rho}| + \eta(\rho)\tilde{S}, \quad (\eta'' \neq 0),$$

and the extension of the kernel is given by the only generator

$$X_3 - 2\tilde{S}\partial_{\tilde{S}},$$

where $\tilde{S} = \mu(S)$. In the final Table 1 this model corresponds to M_{31}.

If $\left(\frac{\rho\eta'''}{\eta''}\right)' = 0$, then $\eta'' = \rho^\nu$, where ν is constant. Further study depends on ν.

If $\nu = -1$, then

$$\eta = \rho\ln\rho. \tag{5.34}$$

Substitution of (5.34) into (4.7) gives

$$2(k_3 - \lambda k_8)(f'\mu - f) = (c_1 - k_8 f), \tag{5.35}$$

where c_1 is a constant of the integration.

Assume that $f'\mu - f = 0$, then $f = q_1\mu$, and equation (5.35) becomes

$$q_1 k_8 \mu = c_1.$$

Because $\mu' \neq 0$, one obtains $q_1 k_8 = 0$ and $c_1 = 0$. If $q_1 = 0$, then

$$W(\rho, \dot{\rho}, \tilde{S}) = q_0 \dot{\rho} \rho^{\lambda} \ln|\dot{\rho}| + \tilde{S} \rho \ln \rho,$$

and the extension of the kernel is given by the generators

$$X_3 - 2\tilde{S}\partial_{\tilde{S}}, \; -\lambda X_1 + X_8 + 2\lambda \tilde{S}\partial_{\tilde{S}},$$

where $\tilde{S} = \mu(S)$. In the final Table 1 this model corresponds to M_{32}. If $q_1 \neq 0$, then $k_8 = 0$. Thus,

$$W(\rho, \dot{\rho}, \tilde{S}) = q_0 \dot{\rho} \rho^{\lambda} \ln|\dot{\rho}| + \tilde{S}(\rho \ln \rho + q_1), \;\; (q_1 \neq 0),$$

and the extension is given by the only generator

$$X_3 - 2\tilde{S}\partial_{\tilde{S}}.$$

In the final Table 1 this model corresponds to M_{33}.

If $f'\mu - f \neq 0$, then

$$k_3 = \lambda k_8 + \frac{c_1 - k_8 f}{2(f'\mu - f)}. \tag{5.36}$$

Differentiating the last equation with respect to μ, one gets

$$\left(\frac{c_1 - k_8 f}{f'\mu - f}\right)' = 0$$

or

$$c_0(f'\mu - f) = c_1 - k_8 f,$$

where c_0 is constant. Notice that if $c_0 = 0$, then an extension of the kernel only occurs for $k_8 \neq 0$. This means that $f = const$ which is without loss of generality can be assumed $f = 0$, and then $f'\mu - f = 0$. Hence one has to assume that $c_0 \neq 0$. This implies that

$$f'\mu - \alpha f = q_3,$$

where $k_8 = c_0(1-\alpha)$ and $c_1 = c_0 q_3$. Notice that by virtue of the equivalence transformation corresponding to the generator X_7^e one can assume that $\alpha q_3 = 0$. We also note that for $\alpha = 1$ one obtains $k_8 = q_3 = 0$, which prohibits an extension of the kernel. Hence, $\alpha \neq 1$. The extension of the kernel of admitted Lie algebras is given by the only generator

$$2(1-\alpha)(X_8 - \lambda X_1 + \lambda X_3) + X_3 - 2\tilde{S}\partial_{\tilde{S}},$$

where

$$W(\rho, \dot{\rho}, \tilde{S}) = q_0 \dot{\rho}\rho^{\lambda} \ln|\dot{\rho}| + \tilde{S}\rho \ln\rho + f(\tilde{S}),$$

$\tilde{S} = \mu(S)$ and

$$f(\tilde{S}) = \begin{cases} q_2 \ln(\tilde{S}), & \alpha = 0; \\ q_2 \tilde{S}^{\alpha}, & \alpha(\alpha - 1) \neq 0. \end{cases}$$

In the final Table 1 these models correspond to M_{34} and M_{35}.

If $\nu = -2$, then $h = \mu(S)\ln\rho + f(\mu(S))$. Integrating equation (4.7), one has

$$(2k_3 - (2\lambda + 1)k_8)(\mu f' - f) - \mu k_8 = c_1, \qquad (5.37)$$

where c_1 is a constant of the integration. If $f'\mu - f = 0$ or $f = q_1\mu$, then $k_8 = 0$, and $c_1 = 0$, and

$$W(\rho, \dot{\rho}, \tilde{S}) = q_0 \dot{\rho}\rho^{\lambda} \ln|\dot{\rho}| + \tilde{S}(\ln\rho + q_1). \qquad (5.38)$$

The extension of the kernel in this case is given by the only generator

$$X_3 - 2\tilde{S}\partial_{\tilde{S}}.$$

In the final Table 1 this model corresponds to M_{36}. If $f'\mu - f \neq 0$, then

$$2k_3 = (2\lambda + 1)k_8 + \frac{c_1 + \mu k_8}{\mu f' - f},$$

and, hence,

$$\left(\frac{c_1 + k_8\mu}{\mu f' - f}\right)' = 0$$

or

$$c_0(\mu f' - f) = c_1 + k_8\mu,$$

where c_0 is constant. Notice that if $c_0 = 0$, then $k_8 = 0$, and there is no an extension of the kernel of admitted Lie algebras. Hence, $c_0 \neq 0$, and

$$f'\mu - f = q_3 + \alpha\mu,$$

where $k_8 = c_0\alpha$ and $c_1 = c_0 q_3$. The general solution of the last equation is

$$f = \alpha\mu\ln(\mu) + q_1\mu - q_3.$$

Thus, the extension of the kernel of admitted Lie algebras is given by the generator

$$-2\lambda\alpha X_1 + (2\alpha\lambda + 1)X_3 + 2\alpha X_8 + 2(\alpha - 1)\tilde{S}\partial_{\tilde{S}},$$

where

$$W(\rho, \dot{\rho}, \tilde{S}) = q_0\dot{\rho}\rho^{\lambda}\ln|\dot{\rho}| + \tilde{S}(\ln\rho + q_1) + \alpha\tilde{S}\ln(\tilde{S}),$$

and $\tilde{S} = \mu(S)$. Notice also that the previous case (5.38) is included in the present case by setting $\alpha = 0$. In the final Table 1 this model corresponds to M_{37}.

Let $(\nu + 1)(\nu + 2) \neq 0$, then $h = \rho^{\nu+2}\mu(S) + f(\mu(S))$, and equation (4.7) gives

$$(2k_3 - (2\lambda - \nu - 1)k_8)(\mu f' - f) + (\nu + 2)fk_8 = c_1. \tag{5.39}$$

If $f'\mu - f = 0$ or $\mu_2 = q_1\mu$, then

$$(\nu + 2)\mu q_1 k_8 = c_1.$$

Because $(\nu+2)\mu' \neq 0$, one obtains that $q_1 k_8 = 0$ and $c_1 = 0$. If $q_1 = 0$, then

$$W(\rho, \dot{\rho}, \tilde{S}) = q_0 \dot{\rho}\rho^{\lambda} \ln|\dot{\rho}| + \tilde{S}\rho^{\nu+2},$$

and the extension of the kernel is given by the generators

$$X_3 - 2\tilde{S}\partial_{\tilde{S}}, \ \ -\lambda X_1 + X_8 + (2\lambda - \nu - 1)\tilde{S}\partial_{\tilde{S}},$$

where $\tilde{S} = \mu(S)$. In the final Table 1 this model corresponds to M_{38}.

If $q_1 \neq 0$, then $k_8 = 0$. Thus

$$W(\rho, \dot{\rho}, \tilde{S}) = q_0 \dot{\rho}\rho^{\lambda} \ln|\dot{\rho}| + \tilde{S}(\rho^{\nu+2} + q_1), \ \ (q_1 \neq 0),$$

and the extension of the kernel is given by the only generator

$$X_3 - 2\tilde{S}\partial_{\tilde{S}}.$$

In the final Table 1 this model corresponds to M_{39}.

If $\mu f' - f \neq 0$, then

$$2k_3 = (2\lambda - \nu - 1)k_8 + \frac{c_1 - (\nu+2)fk_8}{\mu f' - f},$$

and, hence,

$$\frac{c_1 - (\nu+2)fk_8}{\mu f' - f} = c_0,$$

where c_0 is constant. Notice that if $c_0 = 0$, then an extension of the kernel only occurs for $f = const$, whereas by virtue of the equivalence transformation corresponding to the generator X_7^e one can assume that $f = 0$, and then $f'\mu - f = 0$. Hence, $c_0 \neq 0$, and

$$f'\mu - \alpha f = q_2,$$

where

$$c_1 = c_0 q_2, \ \ k_8 = c_0 \frac{1-\alpha}{\nu+2}.$$

Here, as in the previous case, one has to require that $\alpha \neq 1$. Hence,

$$W(\rho, \dot{\rho}, \tilde{S}) = q_0 \dot{\rho} \rho^{\lambda} \ln |\dot{\rho}| + \tilde{S} \rho^{\nu+2} + f(\tilde{S}),$$

and the admitted generator is

$$2\lambda(\alpha - 1)X_1 + (\alpha(\nu + 1 - 2\lambda) + 2\lambda + 1)X_3 + 2(1 - \alpha)X_8 - 2(\nu + 2)\tilde{S}\partial_{\tilde{S}},$$

where

$$f = \begin{cases} q_2 \ln(\tilde{S}), & \alpha = 0; \\ q_2 \tilde{S}^{\alpha}, & \alpha(\alpha - 1) \neq 0. \end{cases}$$

In the final Table 1 these models correspond to M_{40} and M_{41}.

Returning to (5.31), if $\left(\frac{h_{\rho\rho}}{h_{S\rho\rho}}\right)_{\rho} \neq 0$, then equation (5.31) gives

$$2k_3 = -k_8 \left(\frac{\rho h_{\rho\rho\rho} - h_{\rho\rho}(2\lambda - 1)}{h_{S\rho\rho}}\right)_{\rho} / \left(\frac{h_{\rho\rho}}{h_{S\rho\rho}}\right)_{\rho}. \tag{5.40}$$

Thus,

$$\left(\frac{\rho h_{\rho\rho\rho} - h_{\rho\rho}(2\lambda - 1)}{h_{S\rho\rho}}\right)_{\rho} / \left(\frac{h_{\rho\rho}}{h_{S\rho\rho}}\right)_{\rho} = const$$

or

$$\rho h_{\rho\rho\rho} - H h_{S\rho\rho} = k_0 h_{\rho\rho}$$

where k_0 is constant and $H = H(S)$ is some function. Notice that for $H = 0$ one has the contradiction $\left(\frac{h_{\rho\rho}}{h_{S\rho\rho}}\right)_{\rho} = 0$. Hence, $H(S) \neq 0$. The general solution of the last equation (up to an equivalence transformation) is

$$h(\rho, S) = \rho^{\nu} g(\rho\mu(S)) + f(\mu(S)), \tag{5.41}$$

where $\mu' \neq 0$. Equation (4.7) becomes

$$\mu f'' + (\nu + 1)f' = 0. \tag{5.42}$$

Thus,

$$W(\rho, \dot{\rho}, \tilde{S}) = q_0 \rho^{\lambda} \dot{\rho} \ln |\dot{\rho}| + \rho^{\nu} g(\rho\tilde{S}) + f(\tilde{S}),$$

and the extension is given by the only generator

$$-2\lambda X_1 + (2\lambda - \nu + 1)X_3 + 2X_8 - 2\tilde{S}\partial_{\tilde{S}}.$$

Here $\tilde{S} = \mu(S)$, and

$$f(\tilde{S}) = \begin{cases} q_2 \ln(\tilde{S}), & \nu = 0; \\ q_2 \tilde{S}^{-\nu}, & \nu \neq 0. \end{cases}$$

In the final Table 1 these models correspond to M_{42} and M_{43}.

Case $k = -2$.

Integrating (5.24), one obtains

$$\phi = q_0 \rho^{\lambda} \ln|\rho|, \quad (q_0 \neq 0). \tag{5.43}$$

Substituting (5.43) into (4.3), one gets

$$k_1 = -k_3 + k_8 \frac{1-\lambda}{2}.$$

Equation (4.5) becomes

$$2h_{S\rho\rho}\zeta^S = 2q_0 k_3 \rho^{\lambda-2}\lambda(\lambda - 1) - k_8(2\rho h_{\rho\rho\rho} - 2(\lambda - 2)h_{\rho\rho} + q_0\rho^{\lambda-2}\lambda(\lambda^2 - 1)). \tag{5.44}$$

Assuming that $h_{S\rho\rho} = 0$ or

$$h(\rho, S) = \eta(\rho) + \mu(S), \quad (\mu' \neq 0),$$

equation (4.7) and (4.5) are reduced to the equations, respectively,

$$\zeta^S = \frac{k_8\lambda\mu + c_0}{\mu'}, \tag{5.45}$$

$$q_0\lambda(\lambda - 1)(2k_3 - k_8(\lambda + 1)) = 2k_8\rho^{2-\lambda}(\eta'''\rho - \eta''(\lambda - 2)), \tag{5.46}$$

where c_0 is the constant of integration.

Let $\lambda(\lambda-1)\neq 0$. Equation (5.46) gives

$$k_3 = k_8\left(\frac{\lambda+1}{2}+\frac{(\eta'''\rho-\eta''(\lambda-2))}{q_0\lambda(\lambda-1)\rho^{\lambda-2}}\right),$$

Differentiating this equation with respect to ρ, one has

$$k_8\left(\rho^{2-\lambda}(\eta'''\rho-\eta''(\lambda-2))\right)_\rho = 0.$$

If $\left(\rho^{2-\lambda}(\eta'''\rho-\eta''(\lambda-2))\right)_\rho = 0$, then $\eta'' = \rho^{\lambda-2}(\tilde{q}_1 + q_0\lambda(\lambda-1)\nu\ln(\rho))$ or, by virtue of equivalence transformations,

$$\eta = \rho^\lambda(q_1 + q_0\nu\ln(\rho)).$$

Here ν and q_1 are constant. Thus,

$$W(\rho,\dot{\rho},\tilde{S}) = \rho^\lambda(q_1 + q_0\ln(\rho^\nu|\dot{\rho}|)) + \tilde{S},$$

and the extension of the kernel is given by the generators

$$-2(\lambda+\nu)X_1 + (\lambda+2\nu+1)X_3 + 2X_8 + 2\lambda\tilde{S}\partial_{\tilde{S}},\ \partial_{\tilde{S}}.$$

In the final Table 1 this model corresponds to M_{44}. If $\left(\rho^{2-\lambda}(\eta'''\rho-\eta''(\lambda-2))\right)_\rho \neq 0$, then $k_8 = 0$. Thus,

$$W(\rho,\dot{\rho},\tilde{S}) = q_0\rho^\lambda\ln|\dot{\rho}| + \eta(\rho) + \tilde{S},$$

and the extension of the kernel is given by the only generator

$$\partial_{\tilde{S}}.$$

In the final Table 1 this model corresponds to M_{45}.

Let $\lambda(\lambda-1) = 0$. Equation (5.46) becomes

$$k_8(\eta'''\rho-\eta''(\lambda-2)) = 0.$$

If $\eta'' \neq q_1\rho^{(\lambda-2)}$, then $k_8 = 0$. Thus

$$W(\rho, \dot{\rho}, \tilde{S}) = q_0\rho^\lambda \ln|\dot{\rho}| + \eta(\rho) + \tilde{S},$$

and the extension of the kernel is given by the generators

$$-X_1 + X_3,\ \partial_{\tilde{S}}.$$

In the final Table 1 this model corresponds to M_{46}. If $\eta'' = q_1\rho^{(\lambda-2)}$ or

$$W(\rho, \dot{\rho}, \tilde{S}) = q_0\rho^\lambda \ln(|\dot{\rho}|\rho^\nu) + \tilde{S}, \quad (\lambda(\lambda-1) = 0),$$

then the extension of the kernel is given by the generators

$$-X_1 + X_3,\ (1-\lambda)X_1 + 2X_8 + 2\lambda\tilde{S}\partial_{\tilde{S}},\ \partial_{\tilde{S}}.$$

In the final Table 1 this model corresponds to M_{47}.

Assuming that $h_{S\rho\rho} \neq 0$ in equation (5.44), one obtains

$$2\zeta^S = q_0\lambda(\lambda-1)(2k_3 - k_8(\lambda+1))\frac{\rho^{\lambda-2}}{h_{S\rho\rho}} - 2k_8\frac{\rho h_{\rho\rho\rho} - (\lambda-2)h_{\rho\rho}}{h_{S\rho\rho}}. \qquad (5.47)$$

Differentiating the last equation with respect to ρ, one gets

$$q_0\lambda(\lambda-1)(2k_3 - k_8(\lambda+1))\left(\frac{\rho^{\lambda-2}}{h_{S\rho\rho}}\right)_\rho = 2k_8\left(\frac{\rho h_{\rho\rho\rho}}{h_{S\rho\rho}} - \frac{(\lambda-2)h_{\rho\rho}}{h_{S\rho\rho}}\right)_\rho. \qquad (5.48)$$

If $\lambda(\lambda-1) = 0$, then equation (5.48) becomes

$$k_8\left(\frac{\rho h_{\rho\rho\rho}}{h_{S\rho\rho}} - \frac{(\lambda-2)h_{\rho\rho}}{h_{S\rho\rho}}\right)_\rho = 0.$$

Let

$$\left(\frac{\rho h_{\rho\rho\rho}}{h_{S\rho\rho}} - \frac{(\lambda-2)h_{\rho\rho}}{h_{S\rho\rho}}\right)_\rho = 0,$$

then

$$\rho h_{\rho\rho\rho} + H(S)h_{S\rho\rho} = (\lambda-2)h_{\rho\rho}, \qquad (5.49)$$

where $H = H(S)$ is a function of the integration. A solution of the last equation depends on the function $H(S)$.

Assuming that $H = 0$, one has $\zeta^S = 0$,

$$h(\rho, S) = \mu(S)\rho^\lambda \ln \rho + f(\mu(S)),$$

where $\mu' \neq 0$. Equation (4.7) becomes

$$k_8(\lambda f' + q_0(\lambda - 1)\rho^\lambda) = 0. \tag{5.50}$$

If $\lambda = 0$, then equation (5.50) gives $k_8 = 0$. Thus,

$$W(\rho, \dot{\rho}, \tilde{S}) = q_0 \ln|\dot{\rho}| + \widetilde{S} \ln \rho + f(\widetilde{S}),$$

and the extension of the kernel is given by the only generator

$$X_1 - X_3.$$

If $\lambda = 1$, then equation (5.50) becomes $k_8 f' = 0$. For $f' \neq 0$ one has $k_8 = 0$,

$$W(\rho, \dot{\rho}, \tilde{S}) = \rho(q_0 \ln|\dot{\rho}| + \widetilde{S} \ln \rho) + f(\widetilde{S}), \quad (f' \neq 0),$$

and the extension of the kernel is given by the only generator

$$X_1 - X_3.$$

In the final Table 1 these models correspond to M_{48} and M_{49}. For $f' = 0$ one has

$$W(\rho, \dot{\rho}, \tilde{S}) = \rho(q_0 \ln|\dot{\rho}| + \widetilde{S} \ln \rho),$$

and the extension of the kernel is given by the generators

$$X_1 - X_3, \quad X_8.$$

In the final Table 1 this model corresponds to M_{50}.

Assuming that $H \neq 0$ in (5.49), one obtains

$$h(\rho, S) = \rho^\lambda \phi(\rho\mu(S)) + f(\mu(S)), \tag{5.51}$$

where $\mu' \neq 0$. Substitution of $h(\rho, S)$ into (4.7) gives

$$k_8(\mu f' + \lambda f)' = 0. \tag{5.52}$$

If $(\mu f' + \lambda f)' = 0$ or

$$f = \begin{cases} q_1 \ln(\mu), & \lambda = 0, \\ q_1 \mu^{-1}, & \lambda = 1, \end{cases}$$

then

$$W(\rho, \dot{\rho}, \tilde{S}) = \rho^\lambda (q_0 \ln|\dot{\rho}| + \phi(\rho\widetilde{S})) + f(\widetilde{S}),$$

and the extension of the kernel is given by the generators

$$X_1 - X_3, \quad (1-\lambda)X_1 + 2X_8 - 2\tilde{S}\partial_{\tilde{S}}.$$

In the final Table 1 these models correspond to M_{51} and M_{52}. If $(\mu f' + \lambda f)' \neq 0$, then $k_8 = 0$,

$$W(\rho, \dot{\rho}, \tilde{S}) = \rho^\lambda (q_0 \ln|\dot{\rho}| + \phi(\rho\widetilde{S})) + f(\widetilde{S}),$$

and the extension of the kernel is given by the only generator

$$X_1 - X_3.$$

In the final Table 1 this model corresponds to M_{53}.

Returning to (5.48), let $\lambda(\lambda - 1) \neq 0$. Assume also that

$$\left(\frac{\rho^{\lambda-2}}{h_{S\rho\rho}}\right)_\rho = 0,$$

which means that

$$h(\rho, S) = q_0 \mu(S) \rho^\lambda + \eta(\rho) + f(\mu(S)),$$

where $\mu' \neq 0$. Then equations (5.48) becomes

$$k_8 \left(\rho^{2-\lambda}(\rho\eta''' - (\lambda - 2)\eta''\right)_\rho = 0.$$

If $\left(\rho^{2-\lambda}(\rho\eta''' - (\lambda-2)\eta'')\right)_\rho \neq 0$, then $k_8 = 0$. Substituting into (4.7), one obtains

$$k_3 f'' = 0. \tag{5.53}$$

Since for $k_3 = 0$ there is no extension of the kernel, one has $f'' = 0$. Thus,

$$W(\rho, \dot{\rho}, \tilde{S}) = q_0\rho^\lambda(\ln|\dot{\rho}| + \tilde{S}) + \eta(\rho) + q_1\tilde{S},$$

and the extension of the kernel is given by the only generator

$$X_1 - X_3 - \partial_{\tilde{S}}.$$

In the final Table 1 this model corresponds to M_{54}.

If $\left(\rho^{2-\lambda}(\rho\eta''' - (\lambda-2)\eta'')\right)_\rho = 0$, then $\eta'' = \rho^{\lambda-2}(\widetilde{\nu}\ln\rho + \widetilde{q_1})$, where $\widetilde{\nu}$ and $\widetilde{q_1}$ are constant. Using equivalence transformations, one finds that $\eta = \rho^\lambda(q_0\nu\ln\rho + q_1)$, where $\widetilde{\nu} = q_0\nu\lambda(\lambda-1)$ and $\widetilde{q_1} = q_1\lambda(\lambda-1) + q_0\nu(2\lambda-1)$. In this case

$$k_1 = -(k_3 - k_8\frac{\lambda-1}{2}),\ \ \zeta^S = (2k_3 - k_8(\lambda+2\nu+1))/(2\mu'),$$

and equation (4.7) becomes

$$(2k_3 - k_8(\lambda+2\nu+1))f' - 2k_8\lambda f = \widetilde{q_2},$$

where $\widetilde{q_2}$ is constant. The last equation can be rewritten in the form

$$\alpha f' - lf = \widetilde{q_2},$$

where

$$k_8 = \frac{l}{2\lambda},\ \ k_3 = \frac{\alpha}{2} + \frac{l}{2\lambda}\frac{\lambda+2\nu+1}{2}.$$

Further analysis depends on the constants α and l. Notice that for the existence of an extension of the kernel of admitted Lie algebras, one needs to require that $\alpha^2 + l^2 \neq 0$. Hence, for $\alpha = 0$, one has $l \neq 0$, which means that without loss of generality one can assume that $f = 0$. In the case $f = 0$ one obtains

$$W(\rho, \dot{\rho}, \widetilde{S}) = q_0\rho^\lambda(\ln(|\dot{\rho}|\rho^\nu) + \widetilde{S}) + q_1\rho^\lambda,$$

and the extension of the kernel is given by the generators

$$X_1 - X_3 - \partial_{\tilde{S}},\ \ 2(\lambda-1)X_1 + 2X_8 - (\lambda+2\nu+1)\partial_{\tilde{S}}.$$

In the final Table 1 this model corresponds to M_{55}. For $\alpha \neq 0$, one has

$$f = \begin{cases} q_2\mu, & l = 0; \\ q_2 e^{-\kappa\mu} & l \neq 0, \end{cases}$$

and

$$k_1 = -\frac{\kappa(\lambda+\nu)+\lambda}{2\lambda},\ k_8 = \frac{\kappa}{2\lambda},\ \ k_3 = \frac{1}{2} + \frac{\kappa}{4\lambda}(\lambda+2\nu+1),$$

where $l = \kappa\alpha$ and $q_2 \neq 0$ is constant. Thus, one obtains:

(a) for the function $f(\widetilde{S}) = q_2\widetilde{S}$:

$$W(\rho,\dot{\rho},\tilde{S}) = q_0\rho^\lambda(\ln(|\dot{\rho}|\rho^\nu) + \widetilde{S}) + q_1\rho^\lambda + q_2\widetilde{S},\ \ (q_2 \neq 0),$$

and the extension of the kernel is given by the only generator

$$-X_1 + X_3 + \partial_{\tilde{S}};$$

(b) for the function $f(\widetilde{S}) = q_2e^{\kappa\tilde{S}}$:

$$W(\rho,\dot{\rho},\tilde{S}) = q_0\rho^\lambda(\ln(|\dot{\rho}|\rho^\nu) + \widetilde{S}) + q_1\rho^\lambda + q_2e^{\kappa\widetilde{S}},\ \ (q_2 \neq 0),$$

and the extension of the kernel is given by the only generator

$$-2(\kappa(\lambda+\nu)+\lambda)X_1 + 2\kappa X_8 + (2\lambda + \kappa(\lambda+2\nu+1))X_3 + 2\lambda\partial_{\tilde{S}}.$$

In the final Table 1 these models correspond to M_{56} and M_{57}.

Assume that $\left(\frac{\rho^{\lambda-2}}{h_{S\rho\rho}}\right)_\rho \neq 0$, then from (5.48) one finds

$$k_3 = k_8\left(\frac{\left(\frac{\rho h_{\rho\rho\rho}-(\lambda-2)h_{\rho\rho}}{h_{S\rho\rho}}\right)_\rho}{q_0\lambda(\lambda-1)\left(\frac{\rho^{\lambda-2}}{h_{S\rho\rho}}\right)_\rho} + \frac{\lambda+1}{2}\right). \tag{5.54}$$

Since for $k_8 = 0$ there is no an extension, then

$$\frac{\left(\frac{\rho h_{\rho\rho\rho}}{h_{S\rho\rho}} - \frac{(\lambda-2)h_{\rho\rho}}{h_{S\rho\rho}}\right)_\rho}{\left(\frac{\rho^{\lambda-2}}{h_{S\rho\rho}}\right)_\rho} = const$$

or

$$\rho h_{\rho\rho\rho} + H(S)h_{S\rho\rho} = (\lambda-2)h_{\rho\rho} + \nu\rho^{\lambda-2}, \tag{5.55}$$

where ν is constant and $H(S)$ is some function. Notice that for $H(S) = 0$ one obtains

$$h_{\rho\rho} = (\nu \ln\rho + \mu(S))\rho^{\lambda-2},$$

which leads to the contradiction

$$\left(\frac{\rho^{\lambda-2}}{h_{S\rho\rho}}\right)_\rho = 0.$$

Hence, one has to assume that $H(S) \neq 0$, which gives

$$h_{\rho\rho}(\rho, S) = \rho^{\lambda-2}(\widetilde{\nu}\ln\rho + \widetilde{g}(\rho\mu(S)))$$

or

$$h(\rho, S) = \rho^{\lambda}(\nu\ln\rho + g(\rho\mu(S))) + f(\mu(S)),$$

where $\mu' \neq 0$. Equation (4.7) gives $f = q_2\mu^{-\lambda}$. Thus

$$W(\rho, \dot{\rho}, \tilde{S}) = q_0\rho^{\lambda}\left(\ln(|\dot{\rho}|\rho^{\nu}) + g(\rho\widetilde{S})\right) + q_2\widetilde{S}^{-\lambda},$$

and the extension of the kernel is given by the only generator

$$-2(\lambda+\nu)X_1 + 2X_8 + (\lambda+2\nu+1)X_3 - 2\tilde{S}\partial_{\tilde{S}}.$$

In the final Table 1 this model corresponds to M_{58}.

Case $(k+1)(k+2) \neq 0$

Returning to integration of (5.24), if $(k+2)(k+1) \neq 0$, then one obtains

$$\phi = q_0 \rho^{\lambda} \dot{\rho}^{k+2} \tag{5.56}$$

Substituting (5.56) into (4.3), one has

$$k_3 = -k_1 \frac{k}{2(k+1)} + k_8 \frac{k+\lambda+1}{2(k+1)},$$

and equation (4.5) becomes

$$\zeta^S h_{S\rho\rho} + h_{\rho\rho} \left(k_1 \frac{k+2}{k+1} + k_8 \frac{2k+\lambda+2}{k+1} \right) + k_8 \rho h_{\rho\rho\rho} = 0. \tag{5.57}$$

Assuming that $h_{S\rho\rho} = 0$, one finds

$$h(\rho, S) = \eta(\rho) + \mu(S),$$

where $\mu' \neq 0$. Equation (5.57) becomes

$$\eta'' \left(k_1 \frac{k+2}{k+1} + k_8 \frac{2k+\lambda+2}{k+1} \right) + k_8 \rho \eta''' = 0. \tag{5.58}$$

Let $\eta'' \neq 0$, then

$$k_1 = -k_8 \frac{k+1}{k+2} \left(\frac{2k+\lambda+2}{k+1} + \frac{\rho \eta'''}{\eta''} \right).$$

Differentiating the last equation with respect to ρ, one gets

$$k_8 \left(\frac{\rho \eta'''}{\eta''} \right)' = 0. \tag{5.59}$$

If $\dfrac{\rho \eta'''}{\eta''} = k_0 = const$, then $\eta'' = q_1 \rho^{\nu}$, where $\nu = k_0$. This gives that

$$W(\rho, \dot{\rho}, \tilde{S}) = q_0 \rho^{\lambda} \dot{\rho}^{k+2} + \eta(\rho) + \tilde{S}, \ \ (\eta'' = q_1 \rho^{\nu} \neq 0),$$

and the extension of the kernel is given by the generators

$$-\frac{k+1}{k+2} \left(\frac{2k+\lambda+2}{k+1} + \nu \right) X_1 + \frac{k\nu + 3k + 2\lambda + 2}{2(k+2)} X_3 + X_8 + (\nu+2) \tilde{S} \partial_{\tilde{S}}, \ \partial_{\tilde{S}},$$

where $\eta'' = q_1\rho^\nu$, $\tilde{S} = \mu(S)$ and $q_1 \neq 0$. In the final Table 1 this model corresponds to M_{59}, $(k^2+\lambda^2 \neq 0)$.

If $\left(\frac{\rho\eta'''}{\eta''}\right)' \neq 0$, then $k_8 = 0$,

$$W(\rho,\dot{\rho},\tilde{S}) = q_0\rho^\lambda\dot{\rho}^{k+2} + \eta(\rho) + \tilde{S},$$

and the extension of the kernel is given by the only generator

$$\partial_{\tilde{S}}.$$

In the final Table 1 this model corresponds to M_{60}, $(k^2+\lambda^2 \neq 0)$.

Considering (5.58), let $\eta'' = 0$. Without loss of the generality one can assume that $\eta = 0$. Equation (4.7) gives

$$\zeta^S = -\frac{\mu}{\mu'}\left(k_1\frac{k+2}{k+1} + k_8\frac{\lambda}{k+1}\right) + \frac{c_0}{\mu'}.$$

Thus,

$$W(\rho,\dot{\rho},\tilde{S}) = q_0\rho^\lambda\dot{\rho}^{k+2} + \tilde{S},$$

and the extension of the kernel of admitted Lie algebras is defined by the generators

$$X_1 - \frac{k}{2(k+1)}X_3 - \frac{k+2}{k+1}\tilde{S}\partial_{\tilde{S}},\ \ \frac{k+\lambda+1}{2(k+1)}X_3 + X_8 - \frac{\lambda}{k+1}\tilde{S}\partial_{\tilde{S}},\ \partial_{\tilde{S}}.$$

In the final Table 1 this model corresponds to M_{61}, $(k^2+\lambda^2 \neq 0)$. Returning to (5.57), assume that $h_{S\rho\rho} \neq 0$. Then

$$\zeta^S = -\frac{h_{\rho\rho}}{h_{S\rho\rho}}\left(k_1\frac{k+2}{k+1} + k_8\frac{2k+\lambda+2}{k+1}\right) - k_8\frac{\rho h_{\rho\rho\rho}}{h_{S\rho\rho}}.$$

Differentiating this equation with respect to ρ, one finds

$$\left(\frac{h_{\rho\rho}}{h_{S\rho\rho}}\right)_\rho\left(k_1\frac{k+2}{k+1} + k_8\frac{2k+\lambda+2}{k+1}\right) + k_8\left(\frac{\rho h_{\rho\rho\rho}}{h_{S\rho\rho}}\right)_\rho = 0. \quad (5.60)$$

If $\left(\frac{h_{\rho\rho}}{h_{S\rho\rho}}\right)_\rho \neq 0$, then

$$k_1 = -k_8\frac{k+1}{k+2}\left(\frac{2k+\lambda+2}{k+1} + \frac{\left(\frac{\rho h_{\rho\rho\rho}}{h_{S\rho\rho}}\right)_\rho}{\left(\frac{h_{\rho\rho}}{h_{S\rho\rho}}\right)_\rho}\right).$$

Extension of the kernel occurs only for

$$\frac{\left(\frac{\rho h_{\rho\rho\rho}}{h_{S\rho\rho}}\right)_\rho}{\left(\frac{h_{\rho\rho}}{h_{S\rho\rho}}\right)_\rho} = const,$$

which means that

$$\rho h_{\rho\rho\rho} - H(S)h_{S\rho\rho} = \widetilde{\nu} h_{\rho\rho},$$

where $H(S)$ is some function and ν is constant. Notice that for $H(S) = 0$ one has

$$h_{\rho\rho}(\rho, S) = \rho^{\widetilde{\nu}}\mu(S)$$

which leads to the contradiction $\left(\frac{h_{\rho\rho}}{h_{S\rho\rho}}\right)_\rho = 0$. Hence, $H(S) \neq 0$, and then

$$h_{\rho\rho}(\rho, S) = \rho^{\widetilde{\nu}}\tilde{g}(\rho\mu(S)),$$

or

$$h(\rho, S) = \rho^\nu g(\rho\mu(S)) + f(\mu(S)),$$

where $\mu' \neq 0$ and $(z^{\nu+1}g'(z))'' \neq 0$. Equation (4.7) leads to the condition

$$\mu f' + \nu f = \widetilde{q}_2,$$

where $\widetilde{q}_2$ is constant. The general solution of the last equation depends on ν:

$$f(\mu) = \begin{cases} q_2 \ln(\mu), & \nu = 0, \\ q_2\mu^{-\nu}, & \nu \neq 0. \end{cases}$$

Thus, setting $\tilde{S} = \mu(S)$, one gets

$$W(\rho, \dot{\rho}, \tilde{S}) = q_0\rho^\lambda\dot{\rho}^{k+2} + \rho^\nu g(\rho\tilde{S}) + f(\tilde{S}),$$

and the extension of the kernel of admitted Lie algebras is defined by the generator

$$-\frac{\nu(k+1)+\lambda}{k+2}X_1 + \frac{k(\nu+1)+2\lambda+2}{2(k+2)}X_3 + X_8 - \tilde{S}\partial_{\tilde{S}}.$$

In the final Table 1 these models correspond to M_{62} and M_{63}, $(k^2 + \lambda^2 \neq 0)$.

If $\left(\frac{h_{\rho\rho}}{h_{S\rho\rho}}\right)_{\rho} = 0$, then $h(\rho,S) = \mu(S)(\eta(\rho) + f(\mu(S)))$, where $\eta''\mu' \neq 0$. Equation (5.60) becomes

$$k_8\left(\frac{\rho\eta'''}{\eta''}\right)' = 0.$$

If $\left(\frac{\rho\eta'''}{\eta''}\right)' \neq 0$, then $k_8 = 0$, equation (4.7) leads to the equation

$$\mu f'' + 2f' = 0.$$

A solution of the last equation is $f(\mu) = c_1/\mu + c_0$, where c_0 and c_1 are constant. Without loss of generality, one can assume that $c_1 = c_0 = 0$. Thus,

$$k_3 = -k_1\frac{k}{2(k+1)}, \quad \zeta^S = -k_1\frac{k+2}{k+1}\frac{\mu}{\mu'},$$

and

$$W(\rho,\dot{\rho},\tilde{S}) = q_0\rho^{\lambda}\dot{\rho}^{k+2} + \tilde{S}\eta(\rho).$$

The extension of the kernel consists of the generator

$$2(k+1)X_1 - kX_3 - 2(k+2)\tilde{S}\partial_{\tilde{S}}.$$

In the final Table 1 this model corresponds to M_{64}, $(k^2 + \lambda^2 \neq 0)$.

If $\frac{\rho\eta'''}{\eta''} = k_0 = const$, then $\eta'' = \tilde{q}_1\rho^{\nu-2}$, where $\nu = 2(k_0+1)$. One can choose the function $\eta(\rho)$ as follows

$$\eta = \begin{cases} \ln(\rho), & \nu = 0, \\ \rho\ln(\rho), & \nu = 1, \\ \rho^{\nu}, & \nu(\nu-1) \neq 0. \end{cases}$$

This reduces equation (4.7) to the equations

$$\begin{aligned} \nu = 0: &\quad a\mu f' = b + \tilde{q}_2\mu^{-1}, \\ \nu = 1: &\quad a\mu f' + bf = \tilde{q}_2\mu^{-1}, \\ \nu(\nu-1) \neq 0: &\quad a\mu f' + \nu bf = \tilde{q}_2\mu^{-1}. \end{aligned} \tag{5.61}$$

where $a = k_1(k+2) + k_8(\lambda + \nu(k+1))$, $b = k_8(k+1)$ and $\tilde{q}_2$ is constant. Notice that the condition $a^2 + b^2 = 0$ leads to the relations $k_1 = 0$ and $k_8 = 0$. These conditions do not allow an extension of the kernel of admitted Lie algebras. Hence, one has to assume that $a^2 + b^2 \neq 0$.

Let us consider the case $\nu = 0$, where $\eta = \ln(\rho)$. In this case $a \neq 0$, because otherwise $b = 0$. Using equivalence transformations, the general solution of equation $(5.61)_{\nu=0}$ has the representation:

$$f = \beta \ln(\mu) + q_2,$$

where β and q_2 are constant. Substituting the representation of the function $f(\mu)$ into equation $(5.61)_{\nu=0}$, one finds that $\beta a = b$ and $\widetilde{q}_2 = 0$. Therefore,

$$k_1 = a\frac{k+1-\lambda\beta}{(k+1)(k+2)}, \quad k_3 = a\frac{\beta(k+2\lambda+2)-k}{2(k+1)(k+2)}, \quad k_8 = a\frac{\beta}{k+1},$$

and

$$W = q_0\rho^\lambda \dot{\rho}^{k+2} + \tilde{S}\left(\ln\left(\rho\tilde{S}^\beta\right) + q_2\right),$$

where $\tilde{S} = \mu(S)$. The extension of the kernel of admitted Lie algebras is defined by the only generator

$$\frac{k+1-\beta\lambda}{k+2}X_1 + \frac{\beta(k+2\lambda+2)-k}{2(k+2)}X_3 + \beta X_8 - \tilde{S}\partial_{\tilde{S}}.$$

In the final Table 1 this model corresponds to M_{65}, $(k^2 + \lambda^2 \neq 0)$. In other two cases $\nu = 1$ and $\nu(\nu - 1) \neq 0$ one has to solve the equation

$$a\mu f' + \nu b f = \widetilde{q}_2\mu^{-1}, \quad (\nu \neq 0). \qquad (5.62)$$

By virtue of equivalence transformations the function f is equivalent to the function $\tilde{f} = f - r\mu^{-1}$, where r is constant. The change $f = \tilde{f} + r\mu^{-1}$ reduces equation (5.62) to the equation

$$a\mu\tilde{f}' + \nu b\tilde{f} = (\widetilde{q}_2 + (a - \nu b)r)\mu^{-1}.$$

This means that for $a - \nu b \neq 0$ one can assume in (5.62) that $\tilde{q}_2 = 0$. Therefore the analysis of solutions of equation (5.62) is reduced to the study of solutions of either the homogeneous equation

$$a\mu f' + \nu b f = 0, \tag{5.63}$$

or the nonhomogeneous equation

$$\mu f' + f = q_2\mu^{-1}, \quad (q_2 \neq 0). \tag{5.64}$$

The function $f = 0$ is the trivial solution of equation (5.63). In this case k_1 and k_3 are arbitrary. Thus

$$W(\rho, \dot{\rho}, \tilde{S}) = q_0\rho^{\lambda}\dot{\rho}^{k+2} + \tilde{S}\eta(\rho),$$

and the extension of the kernel consists of the generators

$$2(k+1)X_1 - kX_3 - 2(k+2)\tilde{S}\partial_{\tilde{S}}, \quad (k+\lambda+1)X_3 + 2(k+1)X_8 - 2(\lambda+\nu(k+1))\tilde{S}\partial_{\tilde{S}}).$$

Here $\tilde{S} = \mu(S)$. In the final Table 1 these models correspond to M_{66} and M_{70}, $(k^2 + \lambda^2 \neq 0)$.

The only nontrivial solution of equation (5.63) has the representation

$$f(\mu) = q_2\mu^{\beta}, \quad (q_2 \neq 0, \ \beta \neq -1).$$

Substituting the representation into equation (5.63), it becomes

$$\beta(k_1(k+2) + k_8(\lambda + \nu(k+1))) + k_8\nu(k+1) = 0. \tag{5.65}$$

If $\beta = 0$, then $k_8 = 0$, and

$$W(\rho, \dot{\rho}, \tilde{S}) = q_0\rho^{\lambda}\dot{\rho}^{k+2} + \tilde{S}(\eta(\rho) + q_2),$$

with the extension

$$2(k+1)X_1 - kX_3 - 2(k+2)\tilde{S}\partial_{\tilde{S}}.$$

In the final Table 1 these models correspond to M_{67} and M_{71}, $(k^2+\lambda^2 \neq 0)$. If $\beta \neq 0$, then equation (5.65) gives

$$k_1 = -k_8\frac{\beta(\lambda+\nu(k+1))+\nu(k+1)}{\beta(k+2)}.$$

Thus,

$$k_3 = k_8\frac{\beta(k\nu+k+2\lambda+2)+k\nu}{2(k+2)\beta}, \quad \zeta^S = k_8\frac{\nu}{\beta}\frac{\mu}{\mu'},$$

and the potential function is

$$W(\rho,\dot{\rho},S) = q_0\rho^\lambda\dot{\rho}^{k+2} + \tilde{S}(\eta(\rho)+q_2\tilde{S}^\beta), \quad (q_2\beta(\beta+1)\neq 0).$$

The extension of the kernel of admitted Lie algebras is defined by the only generator

$$2\frac{\beta(\lambda+\nu(k+1))+\nu(k+1)}{(k+2)}(X_3-X_1)-(\beta\nu-\beta+\nu)X_3+2\beta X_8+2\nu\tilde{S}\partial_{\tilde{S}}.$$

In the final Table 1 these models correspond to M_{69} and M_{73}, $(k^2+\lambda^2\neq 0)$.

The representation of the general solution of equation (5.64) is $f = q_2\mu^{-1}\ln(\mu)$. Substituting the representation into equation (5.62), it gives

$$\widetilde{q}_2 = aq_2, \quad a-\nu b = 0.$$

Hence,

$$k_1 = -k_8\frac{\lambda}{k+2}.$$

Thus,

$$W(\rho,\dot{\rho},\tilde{S}) = q_0\rho^\lambda\dot{\rho}^{k+2} + \tilde{S}\eta(\rho) + q_2\ln(\tilde{S}), \quad (q_2\neq 0),$$

and the extension of the kernel is defined by the generator

$$2\lambda(X_3-X_1)+(k+2)(X_3+2X_8-2\nu\tilde{S}\partial_{\tilde{S}}).$$

In the final Table 1 these models correspond to M_{68} and M_{72}, $(k^2+\lambda^2\neq 0)$.

5.0.4 $\dim(Span(V)) = 0$

In this case the vector

$$(\dot\rho\phi_{\dot\rho\dot\rho\dot\rho},\ 2(\dot\rho\phi_{\dot\rho\dot\rho\dot\rho} + \phi_{\dot\rho\dot\rho}),\ -(\dot\rho\phi_{\dot\rho\dot\rho\dot\rho} + \rho\phi_{\dot\rho\dot\rho\rho} + \phi_{\dot\rho\dot\rho}))$$

is constant. This condition implies that

$$\phi = q_0\dot\rho^2.$$

Substituting ϕ into (4.3) and (4.5), one gets, respectively,

$$k_3 = \frac{1}{2}k_8,$$

$$\zeta^S h_{S\rho\rho} + 2k_1 h_{\rho\rho} + k_8(\rho h_{\rho\rho\rho} + 2h_{\rho\rho}) = 0. \tag{5.66}$$

Assume that $h_{S\rho\rho} \neq 0$, then

$$\zeta^S = -2ak_1 - k_8 b, \tag{5.67}$$

where $a = \dfrac{h_{\rho\rho}}{h_{S\rho\rho}}$, $b = \dfrac{\rho h_{\rho\rho\rho} + 2h_{\rho\rho}}{h_{S\rho\rho}}$. Differentiating (5.67) with respect to ρ, one obtains

$$2k_1 a_\rho + k_8 b_\rho = 0. \tag{5.68}$$

If $a_\rho = 0$ then, $h(\rho, S) = \eta(\rho)\mu(S) + f(\mu(S))$, where $\eta''\mu' \neq 0$. Equation (5.68) becomes

$$k_8\left(\frac{\rho\eta'''}{\eta''}\right)' = 0.$$

If $\left(\dfrac{\rho\eta'''}{\eta''}\right)' \neq 0$, then $k_8 = 0$, and equation (4.7) becomes

$$k_1 f'' = 0.$$

Since for $k_1 = 0$ there is no extension of the kernel, without loss of generality one can assume that $f = 0$. Thus,

$$W = \dot\rho^2 q_0 + \eta(\rho)\tilde S,$$

and the extension of the kernel is given by the generator

$$X_1 - 2\tilde{S}\partial_{\tilde{S}},$$

where $\tilde{S} = \mu(S)$. In the final Table 1 this model corresponds to M_{64}, $(k = \lambda = 0)$.

If $\left(\frac{\rho\eta'''}{\eta''}\right)' = 0$ or $\eta'' = \rho^{\nu-2}$. Finding the function $\eta(\rho)$ depends on the value of ν.

Let $\nu(\nu-1) \neq 0$, then $\eta = \rho^\nu$ and equation (4.7) becomes

$$2k_1\mu f'' + \nu k_8 (\mu f'' + f') = 0. \quad (5.69)$$

If $f'' = 0$, then $f = q_1\mu$ and equation (5.69) is reduced to the equation

$$k_8 q_1 = 0.$$

Hence, if $q_1 \neq 0$, then $k_8 = 0$ and

$$W = \rho^2 q_0 + (\rho^\nu + q_1)\tilde{S}, \quad (q_1 \neq 0),$$

the extension of the kernel is given by the generator

$$X_1 - 2\tilde{S}\partial_{\tilde{S}},$$

In the final Table 1 this model corresponds to M_{71}, $(k = \lambda = 0)$.

If $q_1 = 0$, then k_8 is arbitrary, and

$$W = \rho^2 q_0 + \rho^\nu \tilde{S}.$$

The extension of the kernel is given by the generators

$$X_1 - 2\tilde{S}\partial_{\tilde{S}}, \; X_3 + 2X_8 - 2\nu\tilde{S}\partial_{\tilde{S}}.$$

In the final Table 1 this model corresponds to M_{70}, $(k = \lambda = 0)$.

If $f'' \neq 0$, then equation (5.69) gives that

$$\mu f'' - \beta f' = 0, \; (\mu \neq 0),$$

where β is constant and

$$k_1 = -\nu k_8 \frac{(\beta+1)}{2\beta}. \tag{5.70}$$

Thus,

$$W = \dot{\rho}^2 q_0 + \rho^\nu \tilde{S} + f(\tilde{S}),$$

and the extension of the kernel is given by the generator

$$-\nu(\beta+1)X_1 + \beta X_3 + 2\beta X_8 + 2\nu \tilde{S}\partial_{\tilde{S}}, \quad (\beta \neq 0).$$

Here

$$f = \begin{cases} q_1 \ln(\tilde{S}), & \beta = -1, \\ q_1 \tilde{S}^{\beta+1}, & \beta \neq -1. \end{cases}$$

In the final Table 1 these models correspond to M_{72} and M_{73}, $(k = \lambda = 0)$.

For $\nu = 1$ one has $\eta = \rho \ln(\rho)$. Further analysis of this equation is similar to the previous case:

$$W = q_0 \dot{\rho}^2 + \tilde{S}(\rho \ln \rho + q_1), \ (q_1 \neq 0): \quad X_1 - 2\tilde{S}\partial_{\tilde{S}},$$

$$W = q_0 \dot{\rho}^2 + \tilde{S}\rho \ln \rho: \quad X_1 - 2\tilde{S}\partial_{\tilde{S}}, \ X_3 + 2X_8 - 2\lambda \tilde{S}\partial_{\tilde{S}},$$

$$W = q_0 \dot{\rho}^2 + \tilde{S}\rho \ln \rho + f(\tilde{S}): \quad -(k+1)X_1 + kX_3 + 2kX_8 + 2\tilde{S}\partial_{\tilde{S}}, \ (k \neq 0),$$

where

$$f = \begin{cases} q_1 \ln(\tilde{S}), & \beta = -1, \\ q_1 \tilde{S}^{\beta+1}, & \beta \neq -1, \end{cases}$$

and $q_1 \neq 0$. In the final Table 1 these models correspond to M_{67}, M_{66}, M_{68} and M_{69}, $(k = \lambda = 0)$, respectively.

Let $\nu = 0$, then $\eta = \ln(\rho)$, and equation (4.7) becomes

$$k_8 = 2k_1 \mu f''.$$

This equation gives

$$k_1 (\mu f'')' = 0.$$

Since for $k_1 = 0$ there is no extension, one has that $\mu f''$ is constant or after using equivalence transformation, one finds

$$f = \mu(\beta \ln(\mu) + q_2).$$

Thus,

$$W = q_0\dot{\rho}^2 + \tilde{S}(\ln(\rho\tilde{S}^\beta) + q_2) : \quad X_1 + \beta(X_3 + 2X_8) - 2\tilde{S}\partial_{\tilde{S}}.$$

In the final Table 1 this model corresponds to M_{65}, $(k = \lambda = 0)$.

If in equation (5.68) $a_\rho \neq 0$, then there exists a constant ν and a function $H(S)$ such that

$$b - \nu a + H(S) = 0$$

or

$$\rho h_{\rho\rho\rho} + H(S)h_{S\rho\rho} = (\nu - 2)h_{\rho\rho}.$$

Hence,

$$k_1 = \nu k_8/2.$$

Notice that if $H = 0$ then $a_\rho = 0$, hence $H \neq 0$. In this case

$$h = \rho^\nu g(\rho\mu(S)) + f(\mu(S)), \tag{5.71}$$

where $\mu' \neq 0$. Equation (4.7) becomes

$$\mu f'' + (\nu + 1)f' = 0. \tag{5.72}$$

Thus,

$$W = q_0\dot{\rho}^2 + \rho^\nu g(\rho\tilde{S}) + f(\tilde{S}) : \quad -\nu X_1 + X_3 + 2X_8 - 2\tilde{S}\partial_{\tilde{S}}.$$

In the final Table 1 these models correspond to M_{62} and M_{63}, $(k = \lambda = 0)$.

If $h_{S\rho\rho} = 0$, then

$$h = \eta(\rho) + \mu(S),$$

where $\mu' \neq 0$, and equations (4.5) (or (5.66)) and (4.7) become, respectively,

$$2k_1\eta'' + k_8(\rho\eta''' + 2\eta'') = 0. \tag{5.73}$$

$$(\zeta^S \mu')' = -2k_1\mu' \tag{5.74}$$

Equation (5.74) gives

$$\zeta^S = (-2k_1\mu + c_0)/\mu' \tag{5.75}$$

Hence, if $\eta'' = 0$, then one can assume that $\eta = 0$. In this case

$$W = q_0\dot{\rho}^2 + \tilde{S},$$

and the extension of the kernel is given by the generators

$$X_1 - 2\tilde{S}\partial_{\tilde{S}}, \ \ X_3 + 2X_8, \ \ \partial_{\tilde{S}}.$$

In the final Table 1 this model corresponds to M_{61}, $(k = \lambda = 0)$.

If $\eta'' \neq 0$, then equation (5.73) leads to

$$k_1 = -k_8\left(\frac{\rho\eta'''}{2\eta''} + 1\right).$$

This gives that

$$k_8\left(\frac{\rho\eta'''}{\eta''}\right)' = 0.$$

For $\rho\eta''' = \nu\eta''$ one has

$$2k_1 + k_8(\nu + 2) = 0.$$

In this case

$$W = q_0\dot{\rho}^2 + \eta(\rho) + \tilde{S}, \ \ (\eta'' = q_1\rho^\nu),$$

and the extension of the kernel is given by the generators

$$-(\nu + 2)X_1 + X_3 + 2X_8 + 2(\nu + 2)\tilde{S}\partial_{\tilde{S}}, \ \ \partial_{\tilde{S}}.$$

Table 1: Group classification

	$W(\rho,\dot\rho,S)$	Extensions	Remarks
M_1	$q_0\rho^{-3}\dot\rho^2+\rho^3S$	$X_1-2S\partial_S,\ \ X_2,\ \ X_3-X_8$	
M_2	$\rho^{-3}\dot\rho^2S+q_1\rho^3S^k$	$X_2,\ \ X_3-X_8,\ \ X_8-(k+1)X_1+2S\partial_S$	
M_3	$\rho^{-3}\dot\rho^2S+\rho^3\mu(S)$	$X_2,\ \ X_3-X_8$	$\mu'\neq q_1S^k$
M_4	$\rho^{-3}\dot\rho^2S$	$X_1,\ \ X_2,\ \ X_3-X_8,\ \ X_8+2S\partial_S$	
M_5	$\phi(\rho,\dot\rho)+S$	∂_S	
M_6	$\varphi(\rho)\dot\rho^2+S$	$\partial_S,\ \ X_1-2S\partial_S$	
M_7	$\varphi(\rho)\dot\rho^2+\eta(\rho)+S$	∂_S	$\eta''\neq 0$
M_8	$\varphi(\rho)\dot\rho^2+\eta(\rho)S$	$X_1-2S\partial_S$	$\eta''\neq 0$
M_9	$\varphi(\rho)\dot\rho\ln\vert\dot\rho\vert+S$	$X_3-2S\partial_S,\ \ \partial_S$	
M_{10}	$\varphi(\rho)\dot\rho\ln\vert\dot\rho\vert+\eta(\rho)+S$	∂_S	$\eta''\neq 0$
M_{11}	$\varphi(\rho)\dot\rho\ln\vert\dot\rho\vert+\eta(\rho)S$	$X_3-2S\partial_S$	$\eta''\neq 0$
M_{12}	$(q_1\rho+q_0)\ln\vert\dot\rho\vert+\eta(\rho)+S$	$X_3-X_1,\ \ \partial_S$	$q_0^2+q_1^2\neq 0$
M_{13}	$\varphi(\rho)\ln\vert\dot\rho\vert+\eta(\rho)+S$	∂_S	$\varphi''\neq 0$
M_{14}	$(q_1\rho+q_0)\ln\vert\dot\rho\vert+h(\rho,S)$	X_3-X_1	$(q_0^2+q_1^2)h_{\rho\rho S}\neq 0$
M_{15}	$\varphi(\rho)(\ln\vert\dot\rho\vert+S)+\eta(\rho)+q_2S$	$X_3-X_1+\partial_S$	$\varphi''\neq 0$
M_{16}	$\varphi(\rho)\dot\rho^{k+2}+S$	$kX_3-2(k+1)X_1+2(k+2)S\partial_S,\ \ \partial_S$	$k(k+1)(k+2)\neq 0$
M_{17}	$\varphi(\rho)\dot\rho^{k+2}+\eta(\rho)+S$	∂_S	$\eta''\neq 0$
M_{18}	$\varphi(\rho)\dot\rho^{k+2}+\eta(\rho)S$	$kX_3-2(k+1)X_1+2(k+2)S\partial_S$	$\eta''\neq 0$
M_{19}	$\rho^\lambda g(\dot\rho\rho^k)+\eta(\rho)+S$	∂_S	
M_{20}	$g(\dot\rho\rho^k)+q_1\rho^2+S$	$-2kX_1+(2k+1)X_3+2X_8,\ \ \partial_S$	
M_{21}	$\rho g(\dot\rho\rho^k)+q_1\rho\ln\rho+S$	$(k+1)(X_3-X_1)+X_8+S\partial_S,\ \ \partial_S$	
M_{22}	$\rho^\lambda g(\dot\rho\rho^k)+S$	$-2(k+\lambda)X_1+(2k+\lambda+1)X_3$ $+2X_8+2\lambda S\partial_S,\ \ \partial_S$	$\lambda(\lambda-1)\neq 0$
M_{23}	$g(\dot\rho\rho^k)+S\ln\rho$	$-2kX_1+(2k+1)X_3+2X_8$	
M_{24}	$\rho g(\dot\rho\rho^k)+S\rho\ln\rho$	$(k+1)(X_3-X_1)+X_8$	
M_{25}	$\rho^\lambda(g(\dot\rho\rho^k)+S)$	$-2(k+\lambda)X_1+(2k+\lambda+1)X_3+2X_8$	$\lambda(\lambda-1)\neq 0$
M_{26}	$g(\dot\rho\rho^k)+Q(\rho S)+q_1\ln S$	$-2kX_1+(2k+1)X_3+2X_8-2S\partial_S$	
M_{27}	$\rho^\lambda g(\dot\rho\rho^k)+Q(\rho S)$	$-2(k+\lambda)X_1+(2k+\lambda+1)X_3+2X_8-2S\partial_S$	$\lambda\neq 0$
M_{28}	$q_0\dot\rho\rho^\lambda\ln\vert\dot\rho\vert+S$	$X_3-2S\partial_S,\ \ X_8-\lambda X_1+(2\lambda+1)S\partial_S,\ \ \partial_S$	
M_{29}	$q_0\dot\rho\rho^\lambda\ln\vert\dot\rho\vert+\eta(\rho)+S$	$2X_8-2\lambda X_1+(2\lambda-\nu-1)X_3+2(\nu+2)S\partial_S,\ \ \partial_S$	$\eta''=q_1\rho^\nu\neq 0$
M_{30}	$q_0\dot\rho\rho^\lambda\ln\vert\dot\rho\vert+\eta(\rho)+S$	∂_S	$\eta''\neq q_1\rho^\nu,\ \ \eta''\neq 0$
M_{31}	$q_0\dot\rho\rho^\lambda\ln\vert\dot\rho\vert+\eta(\rho)S$	$X_3-2S\partial_S$	$\eta''\neq 0$
M_{32}	$q_0\dot\rho\rho^\lambda\ln\vert\dot\rho\vert+S\rho\ln\rho$	$X_3-2S\partial_S,\ \ -\lambda X_1+X_8+2\lambda S\partial_S$	
M_{33}	$q_0\dot\rho\rho^\lambda\ln\vert\dot\rho\vert+S(\rho\ln\rho+q_1)$	$X_3-2S\partial_S$	$q_1\neq 0$
M_{34}	$q_0\dot\rho\rho^\lambda\ln\vert\dot\rho\vert+S\rho\ln\rho+q_2\ln S$	$2(X_8-\lambda X_1+\lambda X_3)+X_3-2S\partial_S$	$q_2\neq 0$
M_{35}	$q_0\dot\rho\rho^\lambda\ln\vert\dot\rho\vert+S\rho\ln\rho+q_2S^\alpha$	$2(1-\alpha)(X_8-\lambda X_1+\lambda X_3)+X_3-2S\partial_S$	$q_2\alpha(\alpha-1)\neq 0$
M_{36}	$q_0\dot\rho\rho^\lambda\ln\vert\dot\rho\vert+S(\ln\rho+q_1)$	$X_3-2S\partial_S$	
M_{37}	$q_0\dot\rho\rho^\lambda\ln\vert\dot\rho\vert+S(\ln\rho+q_1)+\alpha S\ln S$	$-2\lambda\alpha X_1+(2\alpha\lambda+1)X_3+2\alpha X_8+2(\alpha-1)S\partial_S$	$\alpha\neq 0$
M_{38}	$q_0\dot\rho\rho^\lambda\ln\vert\dot\rho\vert+S\rho^{\nu+2}$	$X_3-2S\partial_S,\ \ -\lambda X_1+X_8+(2\lambda-\nu-1)S\partial_S$	$(\nu+2)(\nu+1)\neq 0$
M_{39}	$q_0\dot\rho\rho^\lambda\ln\vert\dot\rho\vert+S(\rho^{\nu+2}+q_1)$	$X_3-2S\partial_S$	$q_1(\nu+2)(\nu+1)\neq 0$
M_{40}	$q_0\dot\rho\rho^\lambda\ln\vert\dot\rho\vert+S\rho^{\nu+2}+q_1\ln S$	$-2\lambda X_1+(2\lambda+1)X_3+2X_8-2(\nu+2)S\partial_S$	$q_1(\nu+2)(\nu+1)\neq 0$
M_{41}	$q_0\dot\rho\rho^\lambda\ln\vert\dot\rho\vert+S\rho^{\nu+2}+q_2S^\alpha$	$2\lambda(\alpha-1)X_1+(\alpha(\nu+1-2\lambda)+2\lambda+1)X_3$ $+2(1-\alpha)X_8-2(\nu+2)S\partial_S$	$(\nu+2)(\nu+1)\neq 0,$ $q_2\alpha(\alpha-1)\neq 0$
M_{42}	$q_0\dot\rho\rho^\lambda\ln\vert\dot\rho\vert+g(\rho S)+q_2\ln S$	$-2\lambda X_1+(2\lambda+1)X_3+2X_8-2S\partial_S$	
M_{43}	$q_0\dot\rho\rho^\lambda\ln\vert\dot\rho\vert+\rho^\nu g(\rho S)+q_2S^{-\nu}$	$-2\lambda X_1+(2\lambda-\nu+1)X_3+2X_8-2S\partial_S$	$\nu\neq 0$
M_{44}	$\rho^\lambda(q_1+q_0\ln(\rho^\nu\vert\dot\rho\vert))+S$	$-2(\lambda+\nu)X_1+(\lambda+2\nu+1)X_3$ $+2X_8+2\lambda S\partial_S,\ \ \partial_S$	
M_{45}	$q_0\rho^\lambda\ln\vert\dot\rho\vert+\eta(\rho)+S$	∂_S	$\eta\neq\rho^\lambda(q_1+q_2\ln\rho)$
M_{46}	$q_0\rho^\lambda\ln\vert\dot\rho\vert+\eta(\rho)+S$	$X_3-X_1,\ \ \partial_S$	$\eta''\neq q_1\rho^{(\lambda-2)},$ $\lambda(\lambda-1)=0$

Table 2: Continuation of Table 1

N	$W(\rho,\dot\rho,S)$	Extensions	Remarks
M_{47}	$q_0\rho^\lambda \ln(\lvert\dot\rho\rvert\rho^\nu)+S$	$X_3-X_1,\ \ (1-\lambda)X_1+2X_8+2\lambda S\partial_S,\ \ \partial_S$	$\lambda(\lambda-1)=0$
M_{48}	$q_0\ln\lvert\dot\rho\rvert+S\ln\rho+f(S)$	X_3-X_1	
M_{49}	$\rho(q_0\ln\lvert\dot\rho\rvert+S\ln\rho)+f(S)$	X_3-X_1	$f'\neq 0$
M_{50}	$\rho(q_0\ln\lvert\dot\rho\rvert+S\ln\rho)$	$X_3-X_1,\ \ X_8$	
M_{51}	$q_0\ln\lvert\dot\rho\rvert+\phi(\rho S)+q_1\ln S$	$X_3-X_1,\ \ X_1+2X_8-2S\partial_S$	$(z\phi(z)')''\neq 0$
M_{52}	$\rho(q_0\ln\lvert\dot\rho\rvert+\phi(\rho S))+q_1S^{-1}$	$X_3-X_1,\ \ X_8-S\partial_S$	$(z\phi(z)')''\neq 0$
M_{53}	$\rho^\lambda(q_0\ln\lvert\dot\rho\rvert+\phi(\rho S))+f(S)$	X_3-X_1	$\lambda(\lambda-1)=0,$ $(Sf'+\lambda f)'\neq 0,$ $(z\phi(z)')''\neq 0$
M_{54}	$q_0\rho^\lambda(\ln\lvert\dot\rho\rvert+S)+\eta(\rho)+q_1S$	$X_3-X_1+\partial_S$	$\lambda(\lambda-1)\neq 0,$ $\eta''\neq\rho^{\lambda-2}(\nu\ln\rho+q_2)$
M_{55}	$q_0\rho^\lambda(\ln(\lvert\dot\rho\rvert\rho^\nu)+S)+q_1\rho^\lambda$	$X_3-X_1+\partial_S,$ $2(\lambda-1)X_1+2X_8-(\lambda+2\nu+1)\partial_S$	$\lambda(\lambda-1)\neq 0$
M_{56}	$q_0\rho^\lambda(\ln(\lvert\dot\rho\rvert\rho^\nu)+S)+q_1\rho^\lambda+q_2S$	$X_3-X_1+\partial_S$	$q_2\lambda(\lambda-1)\neq 0$
M_{57}	$q_0\rho^\lambda(\ln(\lvert\dot\rho\rvert\rho^\nu)+S)+q_1\rho^\lambda+q_2e^{\kappa S}$	$-2(\kappa(\lambda+\nu)+\lambda)X_1+2\kappa X_8+2\lambda\partial_S$ $+(2\lambda+\kappa(\lambda+2\nu+1))X_3$	$q_2\lambda(\lambda-1)\neq 0$
M_{58}	$q_0\rho^\lambda(\ln(\lvert\dot\rho\rvert\rho^\nu)+g(\rho S))+q_2S^{-\lambda}$	$-2(\lambda+\nu)X_1+(\lambda+2\nu+1)X_3+2X_8-2S\partial_S$	$\lambda(\lambda-1)\neq 0,$ $(z^{\lambda+1}g'(z))''\neq 0$
M_{59}	$q_0\rho^\lambda\dot\rho^{k+2}+\eta(\rho)+S$	$-2(2k+\lambda+2+(k+1)\nu)X_1$ $+(k\nu+3k+2\lambda+2)X_3$ $+2(k+2)(X_8+(\nu+2)S\partial_S,\ \ \partial_S$	$(k+1)(k+2)\neq 0,$ $\eta''=q_1\rho^\nu\neq 0$
M_{60}	$q_0\rho^\lambda\dot\rho^{k+2}+\eta(\rho)+S$	∂_S	$(k+1)(k+2)\neq 0,$ $\eta''\neq q_1\rho^\nu, \eta''\neq 0$
M_{61}	$q_0\rho^\lambda\dot\rho^{k+2}+S$	$2(k+1)X_1-kX_3-2(k+2)S\partial_S,$ $(k+\lambda+1)X_3+2(k+1)X_8-2\lambda S\partial_S,\ \ \partial_S$	$(k+1)(k+2)\neq 0$
M_{62}	$q_0\rho^\lambda\dot\rho^{k+2}+g(\rho S)+q_2\ln S$	$-2\lambda X_1+(k+2\lambda+2)X_3+2(k+2)(X_8-S\partial_S)$	$(k+1)(k+2)\neq 0$ $(zg'(z))''\neq 0$
M_{63}	$q_0\rho^\lambda\dot\rho^{k+2}+\rho^\nu g(\rho S)+q_2S^{-\nu}$	$-2(\nu(k+1)+\lambda)X_1+(k(\nu+1)+2\lambda+2)X_3$ $+2(k+2)(X_8-S\partial_S)$	$\nu(k+1)(k+2)\neq 0$ $(z^{\nu+1}g'(z))''\neq 0$
M_{64}	$q_0\rho^\lambda\dot\rho^{k+2}+S\eta(\rho)$	$2(k+1)X_1-kX_3-2(k+2)S\partial_S$	$(k+1)(k+2)\neq 0,$ $\eta''\neq q_1\rho^\nu,\ \eta''\neq 0$
M_{65}	$q_0\rho^\lambda\dot\rho^{k+2}+S(\ln(\rho S^\beta)+q_2)$	$2(k+1-\beta\lambda)X_1+(\beta(k+2\lambda+2)-k)X_3$ $+2(k+2)(\beta X_8-S\partial_S)$	$(k+1)(k+2)\neq 0$
M_{66}	$q_0\rho^\lambda\dot\rho^{k+2}+\rho\ln(\rho)S$	$2(k+1)X_1-kX_3-2(k+2)S\partial_S,$ $2\lambda(X_3-X_1)+(k+2)(X_3+2X_8-2S\partial_S)$	$(k+1)(k+2)\neq 0$
M_{67}	$q_0\rho^\lambda\dot\rho^{k+2}+S(\rho\ln(\rho)+q_2)$	$2(k+1)X_1-kX_3-2(k+2)S\partial_S$	$q_2\neq 0,$ $(k+1)(k+2)\neq 0$
M_{68}	$q_0\rho^\lambda\dot\rho^{k+2}+S\rho\ln\rho+q_2\ln S$	$2\lambda(X_3-X_1)+(k+2)(X_3+2X_8-2S\partial_S)$	$q_2\neq 0,$ $(k+1)(k+2)\neq 0$
M_{69}	$q_0\rho^\lambda\dot\rho^{k+2}+S(\rho\ln\rho+q_2S^\beta)$	$2(\beta(k+\lambda+1)+k+1)(X_3-X_1)$ $-(k+2)X_3+2(k+2)(\beta X_8+S\partial_S)$	$\beta q_2\neq 0,$ $(k+1)(k+2)\neq 0$
M_{70}	$q_0\rho^\lambda\dot\rho^{k+2}+S\rho^\nu$	$2(k+1)X_1-kX_3-2(k+2)S\partial_S,$ $(k+\lambda+1)X_3+2(k+1)X_8$ $-2(\lambda+\nu(k+1))S\partial_S)$	$\nu\neq 0,$ $(k+1)(k+2)\neq 0$
M_{71}	$q_0\rho^\lambda\dot\rho^{k+2}+S(\rho^\nu+q_2)$	$2(k+1)X_1-kX_3-2(k+2)S\partial_S$	$q_2\neq 0,$ $(k+1)(k+2)\neq 0$
M_{72}	$q_0\rho^\lambda\dot\rho^{k+2}+S\rho^\nu+q_2\ln S$	$2\lambda(X_3-X_1)+(k+2)(X_3+2X_8-2\nu S\partial_S)$	$q_2\neq 0,$ $(k+1)(k+2)\neq 0$
M_{73}	$q_0\rho^\lambda\dot\rho^{k+2}+S(\rho^\nu+q_2S^\beta)$	$2\beta(\lambda+\nu(k+1))+\nu(k+1)(X_3-X_1)$ $-(k+2)(\beta\nu-\beta+\nu)X_3$ $+2(k+2)(\beta X_8+\nu S\partial_S)$	$q_2\beta\neq 0,$ $(k+1)(k+2)\neq 0$

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